CENTRES FOR EXCELLENCE IN MATHS

EDUCATION & TRAINING FOUNDATION

GATESHEAD COLLEGE
EDUCATION WITH EMPLOYMENT EDGE

Maths Reasoning

Developing Mathematical Reasoning; The role of the CPA model in students' progress from standard to Reasoning and Problem-Solving questions.

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About CfEM

Centres for Excellence in Maths (CfEM) is a five-year national improvement programme aimed at delivering sustained improvements in maths outcomes for 16–19-year-olds, up to Level 2, in post-16 settings.

Funded by the Department for Education and delivered by the Education and Training Foundation, the programme is exploring what works for teachers and students, embedding related CPD and good practice, and building networks of maths professionals in colleges.

Summary

This action research project aimed to explore the impact of the Concrete Pictorial and Abstract (CPA) approach on students' reasoning skills development. The initial assessments reflected students' profound difficulty to reason mathematically and problem solve. The absence of valid and efficient solving methods in their answers, revealed the need to apply meaningful teaching approaches and solving strategies that enhance deeper understanding. A CPA model has been applied to support the efforts of two student groups to develop reasoning skills. Both quantitative and qualitative methods were applied to investigate the impact of the CPA model students' reasoning approaches. The findings in Cycle 1 of the research showed a significant improvement on reasoning skills development. Student surveys and work samples showed that bar models were powerful representations. However, students' methods revealed misconceptions on fundamental concepts such as unitizing and so these were further investigated in Cycle 2 of the project. Although students were willing to try new solving methods, they need more time to work on learning gaps and misconceptions that impact their ability to reason mathematically. Further thoughts were generated on the time restrictions that Further Education teachers face regarding the long list of topics that needs to be taught.

Contents

Background

Introduction

Established in 1955, Gateshead College constitutes one of the largest and further education colleges in Northeast of England. Gateshead College delivers courses to address the educational needs of 16-18-year-olds, apprenticeships, higher education courses as well as part-time adult learning programs. The college is currently one of the largest Further Education colleges in Northeast of England. The courses are delivered in six campuses where both academic and vocational programs take place.

College Goals

Gateshead College sets Employment Edge as the central goal of all students through Collaborative Working. The college works with individuals, business, community organisations and agencies to develop programmes and provide placement opportunities to ensure that the Higher Education provision meets. The core values of the college are achievement, excellence, partnership, positivity, purposefulness, and respect. The college is fully committed to equality and diversity.

Gateshead College sets high standards in academic excellence in mathematics. For that purpose, the college continues to be part of the Centre for Excellence in Mathematics program. The college is one of the 21 college centres and administers CfEM training and meetings with 7 partner colleges.

Research Aims

The aim of the action research project was to investigate the impact of a mastery approach method on students' development of mathematical reasoning skills. For that purpose, the model we selected to investigate was the CPA model. CPA stands for Concrete, Pictorial, Abstract representations. The CPA approach aims to help students to develop a deep and understanding of mathematical concepts. In this project, the CPA model was used as a method to support students to develop meaningful and valid problem-solving strategies. Students could use any stage of the CPA model they felt comfortable with (concrete, pictorial, abstract), acknowledging that they came to Further Education with their previous taught methods and possible learning gaps and misconceptions.

Literature Review

After two years of interrupted learning due to the pandemic, students in FE are back to the classrooms facing the challenges of the next day. Students started the year with learning gaps from their previous secondary school years, low confidence, along with an urgent need to dust off communication skills. Based on the Initial Assessments data, administered at Gateshead College, our students had significant difficulty to communicate mathematically their solutions. Students performed significantly lower on Reasoning and Problem-Solving questions to the extent that many of these questions were not even attempted.

The question that consequently emerged from these findings, was how to support our students in developing mathematical reasoning skills. Given our time restrictions to cover all the GCSE topics within one academic year, we chose to apply the Concrete-Pictorial-Abstract (CPA) model to bridge the transition from concrete and pictorial thinking to the abstract level. Moving forward, we reviewed articles available on ERIC, Research Gate and Google Scholar to help us identify the teaching strategies we need to employ in order to support our students' reasoning skills in Mathematics.

The Concrete-Pictorial-Abstract model has its roots in Jerome Bruner's (1966) theory of three modes of representations, originally named as *enactive, symbolic, iconic* representation (Bruner, 1966). The Concrete-Representational -Abstract model (CRA) is an adaptation of Bruner's three modes (Leong, 2015). At the concrete level students manipulate actual concrete objects (like counters, snap cubes) in their learning, while representations refer to images of objects and drawings (Holmes, 2013; Kontaş, 2016). At the abstract level students use only symbols and numbers in mathematical problem solving. In the early '80s, the acronym CPA has been used in Singapore to describe the three stages of Concrete, Pictorial, Abstract level (Leong, 2015). Both terms, CRA and CPA, refer to the facilitation of the transition of students to abstract thinking in maths, via scaffolding.

Research indicates that in earlier school years, concrete and pictorial representations are powerful and effective. The use of certain manipulatives and methods derived from the Mastery approach was significantly effective with students that struggle with mathematics. Morin, et al (2017) report the effectiveness of the use of bar models on word problem-solving as an intervention method (Morin, Watson, Hester, & Raver, 2017). The study involved just a small group of primary school students with learning difficulties in Mathematics. Their findings indicate significant impact on student development and maintenance of strategies and accuracy.

Ruchti and Bennett (2013) provided a class of seventh and eighth grade students with a proportional reasoning task (Bennett, 2013). Students were asked to use pictorial representations to explain and justify their answers to another peer. The students' representations revealed a common misconception; the use of additive rather than multiplicative reasoning to carry out the task. The study highlighted how different visual representations may support students to confront their own misunderstandings and choose the appropriate visual models to effectively represent their solutions. The researchers found that the use of pictorial representations is a powerful tool that students can use to develop reasoning skills. Visual representations lead to deeper understanding as students can further choose the suitable visual model on a case-by-case basis.

Regarding AO2 and AO3 questions, researchers investigated the impact of the CPA approach on understanding, representing, and reasoning on fractions (Purwadi, Sudiarta, &

Suparta, 2019). The study was conducted using an experimental and a control group of third grade students. The researchers applied a mixed method with simultaneous strategies to examine the impact on mathematical conceptual understanding and mathematical representation.
The CPA model was found to have a significant positive impact on students' understanding and accuracy levels. The findings of the study reflect a strong impact on students' confidence in reasoning, communication skills and student motivation. Researchers recommended for more studies to be carried out on a larger group with more materials. They also referred to the difficulties and challenges of applying the CPA model, such as the time required.

Another study aimed to investigate the impact of the CPA model on the reasoning skills of 121 fifth grade students of two different primary schools (Putri, 2020). The study applied a quasi-experiment method using pre- and post-test control and experimental groups, to compare the impact of the conventional learning strategies versus the CPA approaches. The prior mathematical ability (PMA) of the students was taken into consideration by using a PMA test prior to the conduction of the study. Students were classified into low, moderate, and high MPA categories. The data analysis included the impact of the CPA model on all three groups. Researchers found that the implementation of the CPA model improved students' ability to reason mathematically on fractions across student of different prior mathematical ability. The CPA model was found to have a profound impact on the learning activities of primary school students.

Although literature indicates a significant positive impact of CPA model practices and models on students reasoning and problem-solving skills, more research is needed for the further education student population. Our search revealed the difficulty in finding articles investigating the impact of mastery approaches such as the CPA models and strategies on further education. We were mainly able to find articles where research has been conducted in primary and middle school settings. No findings have been identified on the application of concrete manipulatives and pictorial representations in the 16-18 age group.

Given that this age group has been previously taught the topics covered in our GCSE classrooms, our students have been exposed to several teaching styles. Time restrictions have been identified as another factor that impedes and impacts the implementation of the CPA model in the classrooms. As a result, we would like to conduct our research under the above considerations, highlighting the important role of qualitative data in our next steps.

Methods

Research Process

The action research project aimed to investigate the impact of CPA model approach may have on students' performance on reasoning skills questions. Furthermore, the project explored student's responses and views among various representations.

The project took place in two consecutive Cycles. The chronological structure of each cycle was designed based on the stages outlined in the 5-minute research plan (see Appendix 1). The 5-minute plan was completed twice and used to describe the phases of the two Cycles.

In both cycles, the research plan included the use of mixed quantitative and qualitative methods to answer our action research questions. We designed our own assessment tools, surveys, as well as paper-based classroom activities to gather data on students' performance and solving methods. This stage was completed along with the article reading along with administration of the assessments. The reading process also helped us to inform the design of the CPA focused lessons.

During Cycle 1, mixed quantitative and qualitative methods were used aiming to answer our action research questions. The data collected aim to investigate the following question of this project:

-Did the CPA model approach, improve students' reasoning skills?

-What were the representations that students prefer to choose to use to solve the problems on pre and post assessments?

Quantitative methods were used to address the first question. Regarding the second question mixed methods were applied. We used surveys that included multiple choice and open-ended questions. Qualitative data included interviews from eight students.

After completing Cycle 1, the next phase aimed to focus on investigating in depth the student's preference on certain models and methods to solve problems. We wanted to investigate further how individuals used the models that they found helpful in Cycle 1. For that purpose, our original title of the project changed by moving the apostrophe from *students'* to *student's*: Developing Mathematical Reasoning; Apply CPA model to improve student's performance on Reasoning (AO2) and Problem Solving (AO3) questions.

The questions we aimed to address during cycle 2 were the following:

-Did student use effective methods to answer questions on fractions?

-What were the barriers in student's learning experience that impeded the development of valid reasoning methods?

Qualitative data were collected and quantified using coding to analyze students' preferable methods and its effectiveness. A sample of seven students were chosen to participate in follow up video-based interviews during the third week. Student group consisted of students who use visual representations both effectively and ineffectively. The interviews aimed to give an insight on student's thoughts on choosing a solving method. Thus, the research question will focus on individual student's perspective on developing reasoning methods. The same interview questions were used in Sunderland College group using note taking.

Ethical considerations

All students were provided with the support needed during the lessons, as well as the groups from other vocational areas that were not part of the project. Anonymity was protected while we were taking students' work sample work by covering their personal details. For the conduction of video interviews, consent forms have been provided to be singed.

The impact of Covid

During our first meetings with other colleges, there was a great interest on participating in this project. However, since the beginning of Cycle 1, teachers found it difficult to participate in the project, attending meetings, as well as meeting the deadlines due to the conditions related to Covid.

Dissemination

The findings of this project will be shared and discussed with the teachers in Gateshead college during our scheduled Maths meetings. The data will also be communicated with our local network during the presentations scheduled by CfEM on June 24th 2022.

Implementation

Cycle 1

The Setting

Two mixed ability GCSE Maths student groups were selected in early September. The participant students consisted of two Art-Design-Media groups, initially a total of 42 students. The students' learning profiles varied regarding performance and learning needs.

The total number of students that participated in Cycle 1 dropped due to the challenges of the pandemic, as well as for the logistics of students' timetable. A total of twenty students from both groups participated in the Fractions intervention. For the Ratio activity, we collected data from twenty-three students. The groups were taught by the same teacher. Figure 1 shows the four stages that have been completed during Cycle 1.

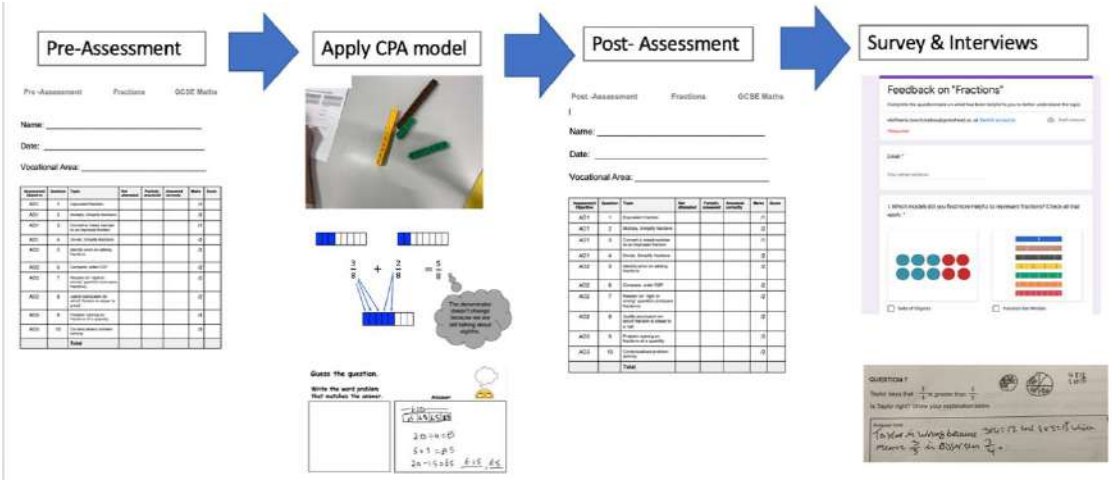


Figure 1. Cycle 1 in chronological order.

Pre-Assessment

During the first phase of cycle 1, students sat a pre- and post-assessment on fractions both fractions and ratios (see Appendix 2). In between the assessments, students attended three-week and two-week lessons on each topic respectively. The fraction assessment consisted of GCSE exam like questions including four recall/use knowledge (AO1), four reasoning (AO2) and two problem-solving questions (AO3) questions. The ratio assessment consisted of five recall/use knowledge (AO1), three reasoning (AO2) and two problem-solving (AO3) GCSE exam like questions. The data has been recorded by question item and grouped by the type of the questions. The data aimed to compare students' performance per question, as well as among recall knowledge, reasoning, and problem-solving questions.

CPA-centred lessons

The lessons were implemented after the completion of the pre-assessment in end of November and early December 2021 for six weeks. The first three weeks students were taught Fractions, followed by the post-Assessment. The following two weeks, we delivered lessons on ratio and proportion.

Both groups attended two 90-minute lessons weekly for three consecutive weeks. The first lesson included a starter activity (revision on various topics), the main lesson of topic and independent work/classwork to practice the skills on the topic. The second lesson begun with a starter activity, followed by an extension of the topic on reasoning and problem-solving questions. The second lesson ended with the end of week topic assessment.

In terms of the resources, students had access to Google Classroom and all resources have been used in the classroom were posted on Google Classroom for students to access them at home. Students had access to maths manipulatives such as snap cubes, fraction bars and fraction circles, as well as to pictorial representations templates (print outs of bar models), displayed on their desks to be used as needed. The lesson presentation included a Google Slide presentation with visual representations of the models that students were encouraged to use during classwork. Also, websites such as *Mathsbot* and T*oyTheater* offering, virtual maths manipulatives, were presented to students and posted on google classroom, as an option to use them for both homework and classwork. Students could use concrete pictorial abstract ways to work on a question, in any order and as needed. Figures 2 and 3 show examples of students' work that depict the transition among the representations. Students that used maths manipulatives as part of their method, they could upload photos of their work on their personal Jamboard slide to show their reasoning. Figure 4 shows a Fraction activity were students need to apply higher thinking skills. Students need to find the question of a problem that has been solved using bar modeling (see Appendix 5).

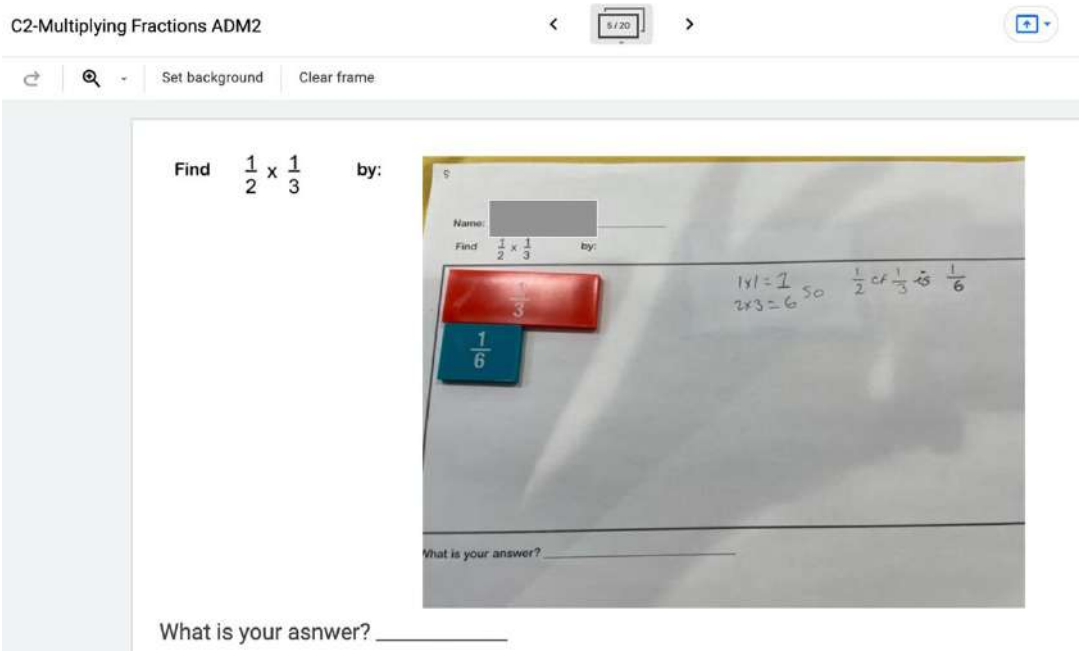


Figure 2. Student's work on Jamboard.

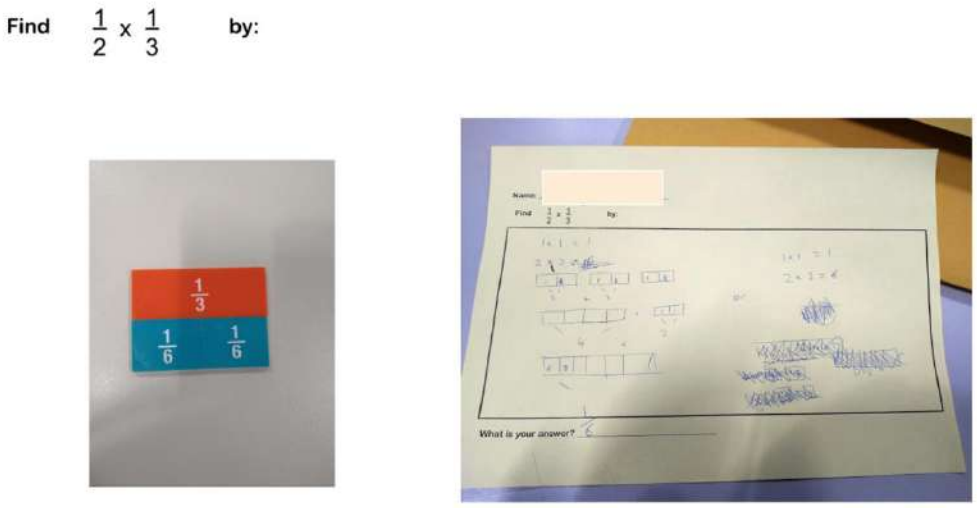


Figure 3. Student's work on Jamboard. Uploaded photos of work, using maths manipulatives and pictorial representations.

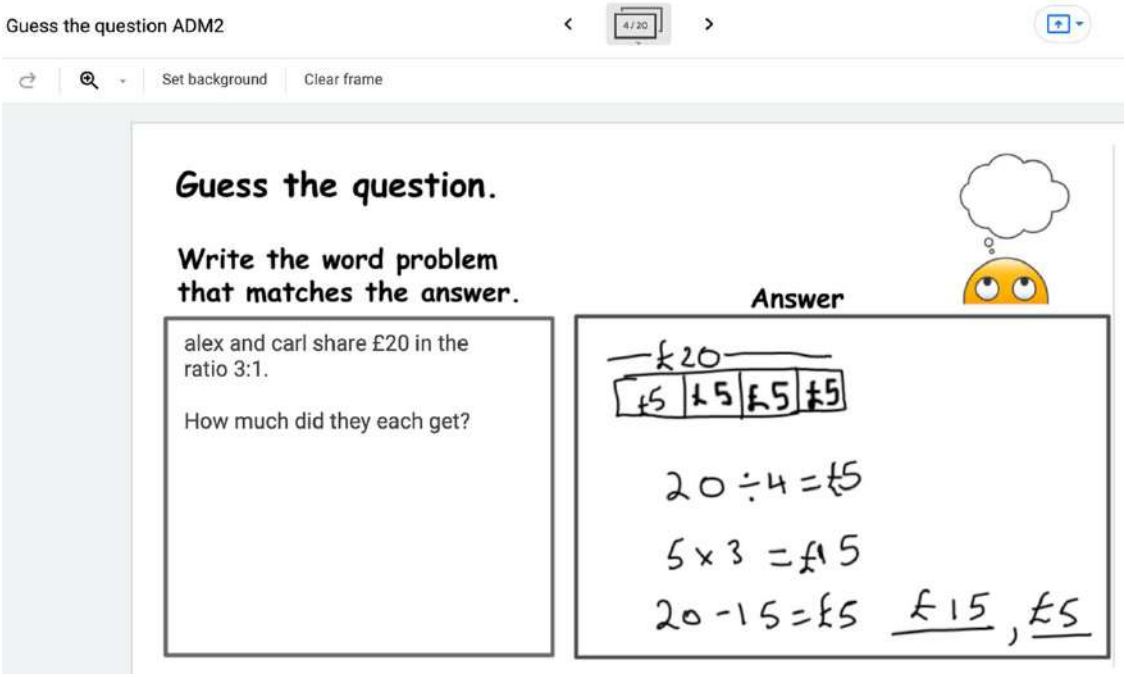


Figure 4. Jamboard Activity on Fractions using the CPA model.

Post-Assessment

When the lessons on Fractions and Ratio were completed, the students were given the same assessment as a post-assessment to monitor and record progress. The Pre- and Post-assessment provided us with quantitative data. The data have been recorded on the same spreadsheet that was used for the Pre-Assessment to be able to compare students' performance before and after the CPA lessons were delivered.

Surveys & Interviews

The students participated two separate surveys: one assessment on Fractions and one on Ratio (see Appendix 6).

The first survey aimed to capture student's responses on their learning experience using representations and various models. Students expressed their thoughts on which model(s) they found useful, such as fraction circles, bar models, sets of objects etc. The survey aimed to capture the models that were defined as more helpful for students when they work with certain questions (i.e., find a fraction of an amount). Also, students were asked to name or describe a "trick" method they have been taught in the past that they can apply successfully.

The ratio survey consisted of three questions. Students were asked about which representation was more helpful to them while working with ratio (sets of objects, tables, bar models, abstract/words & numbers). Students also expressed their thoughts on if their preferred representations have been helpful in various types of ratio questions. They were also asked if they consider representations as a valid method to use during GCSE exams.

Following the completion of the surveys, eight students have been interviewed on their selected solving methods to provide us with more information on the representation they find most efficient for their solving methods.

Mixed quantitative and qualitative methods were used, including data from the assessments and surveys using google forms, as well as qualitative data deriving from students' interviews.

Cycle 2

The setting

The student groups that participated in Cycle 1 at Gateshead College will also participate in Cycle 2. The groups consisted of the same two Art-Design-Media mixed ability groups. The total number of students who participated in further fraction activities was twenty-five. Interviews were performed with a smaller group of seven students to investigate in depth the students' solving methods on their final activity. Six out of seven students completed and submitted the consent forms.

Also, a group of seven students from Sunderland College participated in the last fraction activity. The activity was completed during the starting time of a lesson, and when students have already completed their lessons on both fractions and ratio.

Fraction activities using CPA at Gateshead College

Students worked in six activities on Fractions (representation, equivalence, addition, multiplication, fraction of an amount, division). The activities conducted in six lessons within three weeks. Every week two activities were covered during the first part of each of the two weekly lessons.

The fraction activities have been designed, using GCSE exam like questions. The format of the activities encouraged students to use different ways to model/representation to support their reasoning (see Appendix 7). Students were asked one question (i.e., Show four thirds) and asked to draw as many pictorial ways as they wanted to try (sets of objects, fraction circles, bar models, number lines, fraction squares, words/symbols/numbers). Students were asked to freely select one or more models of their choice. The teacher explained that the

purpose of the activity was to explore ways that would work best for their problem-solving methods, and they shouldn't worry about if their answer is right or wrong or if they try just one type of representation.

The last activity on fractions was designed differently (see Appendix 4). Students were given a question in words (see Figure 5) and were asked to give their reasoning in an answer box. They could choose the model and method of their choice to answer this question.

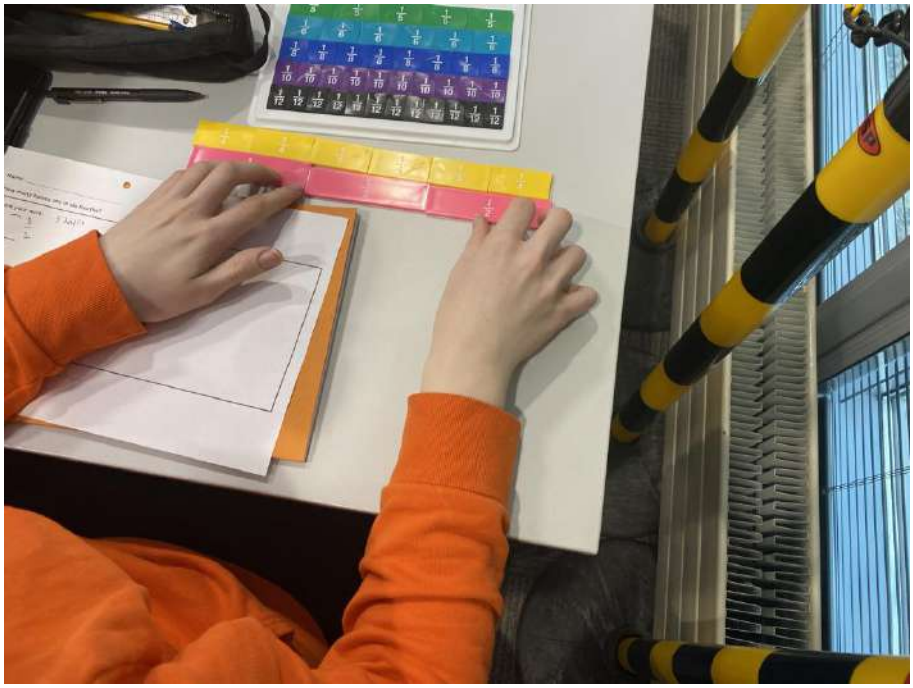


Figure 5. A student is working on a division question:" how many halves in six fourths?" using both the concrete and pictorial method simultaneously.

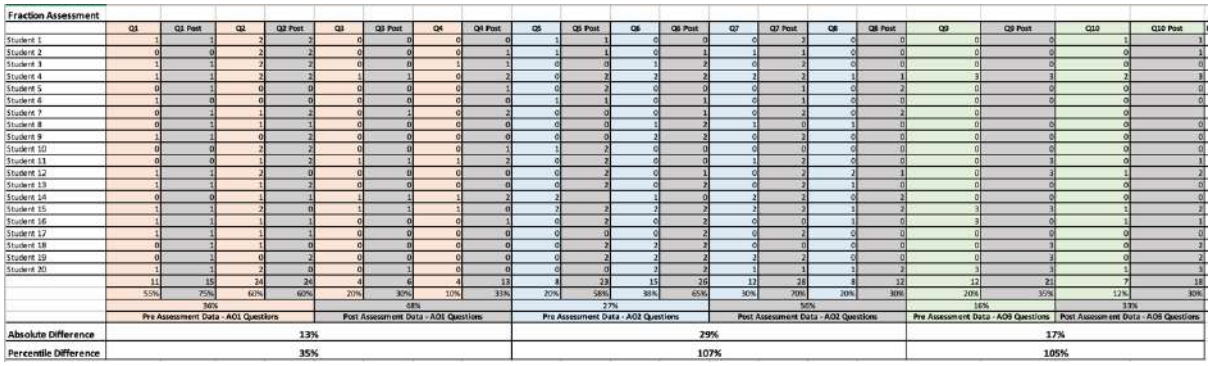
Findings

Cycle 1

Pre- and Post- Assessment on Fractions

The data from both Pre- and Post-Assessment on Fraction assessment showed a significant improvement on students' performance on reasoning and problem-solving questions (see Figure 6). The improvement for both type of questions exceeded the 100%. More precisely:

- Students improved their performance on recall/use knowledge questions by 35%
- Students improved their performance on reasoning questions by 107%
- Students improved their performance on problem solving questions by 105%



Fraction Assessment	Q1	Q1 Post	Q2	Q2 Post	Q3	Q3 Post	Q4	Q4 Post	Q5	Q5 Post	Q6	Q6 Post	Q7	Q7 Post	Q8	Q8 Post	Q9	Q9 Post	Q10	Q10 Post
Student 1	1	1	2	2	0	0	0	0	1	1	0	0	0	3	0	0	0	0	1	1
Student 2	0	0	2	2	0	0	0	1	1	1	0	1	1	1	0	0	0	0	0	1
Student 3	1	1	2	2	0	0	1	1	0	0	1	2	0	2	0	0	0	0	0	0
Student 4	1	1	2	2	1	1	0	2	0	2	2	2	2	2	1	1	3	3	2	3
Student 5	0	1	0	0	0	0	0	1	0	2	0	0	0	1	0	2	0	0	0	0
Student 6	1	0	0	0	0	0	0	0	1	1	0	1	0	1	0	0	0	0	0	0
Student 7	0	1	1	2	0	1	0	2	0	0	0	1	0	2	0	2	0	0	0	0
Student 8	0	1	1	1	0	0	0	0	0	0	1	2	1	0	1	0	0	0	0	0
Student 9	1	1	0	2	0	0	0	0	0	0	2	2	0	2	0	0	0	0	0	0
Student 10	0	0	2	2	0	0	0	1	1	2	0	0	0	0	0	0	0	0	0	0
Student 11	0	0	1	2	1	1	1	2	0	2	0	0	1	2	0	0	0	3	0	1
Student 12	1	1	2	0	0	0	0	0	0	2	0	1	0	2	2	3	0	3	1	2
Student 13	1	1	1	2	0	0	0	0	0	2	0	2	0	2	1	0	0	0	0	0
Student 14	0	0	1	1	1	1	1	2	2		1	0	2	2	0	2	0	0	0	0
Student 15	1	1	2	0	1	1	1	0	2	2	2	2	2	2	1	2	3	3	1	2
Student 16	1	1	1	1	0	0	0	1	0	2	0	2	0	0	1	0	3	0	1	1
Student 17	1	1	1	1	0	0	0	0	0	0	0	2	0	2	0	0	0	0	0	0
Student 18	0	1	1	0	0	0	0	0	0	2	2	3	0	0	0	0	0	3	0	2
Student 19	0	1	0	2	0	0	0	0	0	2	2	2	2	2	0	0	0	3	0	2
Student 20	1	1	2	0	0	1	0	0	0	0	2	2	1	1	1	2	3	3	1	1
	11	15	24	24	4	6	4	13	8	23	15	26	13	28	8	12	12	21	7	18
	55%	75%	60%	60%	20%	30%	10%	33%	20%	58%	38%	65%	30%	70%	20%	30%	20%	35%	12%	30%
		36%				68%				27%				55%				16%		33%
	Pre Assessment Data - AO1 Questions					Post Assessment Data - AO1 Questions				Pre Assessment Data - AO2 Questions				Post Assessment Data - AO2 Questions				Pre Assessment Data - AO3 Questions		Post Assessment Data - AO3 Questions
Absolute Difference	13%								29%								17%			
Percentile Difference	35%								107%								105%			

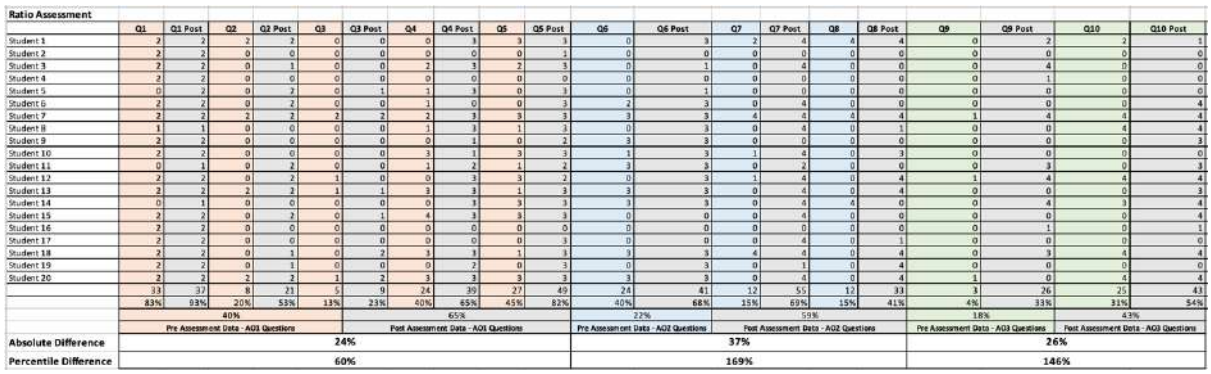
Figure 6. Performance Data on Pre- and Post- Assessment on Fractions.

The overall performance of students on the assessments was lower though than expected. Analysing students samples we observed that students found difficult to represent their solutions especially on questions that involve the representation of a "whole".

Pre- and Post- Assessment on Ratio

The overall performance of students on the assessments was higher on Ratio Assessment compared to Fraction Assessment in all three types of questions (see Figure 7). Students showed again greater improvement on Reasoning and Problem-Solving questions.

- Students improved their performance on recall/use knowledge questions by 60%
- Students improved their performance on reasoning questions by 169%
- Students improved their performance on problem solving questions by 146%



Ratio Assessment	Q1	Q1 Post	Q2	Q2 Post	Q3	Q3 Post	Q4	Q4 Post	Q5	Q5 Post	Q6	Q6 Post	Q7	Q7 Post	Q8	Q8 Post	Q9	Q9 Post	Q10	Q10 Post
Student 1	2	2	2	2	0	0	0	3	3	3	0	3	2	4	4	4	0	2	2	1
Student 2	2	2	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
Student 3	2	2	0	1	0	0	2	3	2	3	0	1	0	4	0	0	0	4	0	0
Student 4	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
Student 5	0	2	0	2	0	1	1	3	0	3	0	1	0	0	0	0	0	0	0	0
Student 6	2	2	0	2	0	0	1	0	0	3	2	3	0	4	0	0	0	0	0	4
Student 7	2	2	2	2	2	2	2	3	3	3	3	3	4	4	4	4	1	4	4	4
Student 8	1	1	0	0	0	0	1	3	1	3	0	3	0	4	0	1	0	0	4	4
Student 9	2	2	0	0	0	0	0	1	0	2	3	3	0	0	0	0	0	0	0	3
Student 10	2	2	0	0	0	0	1	3	3	3	1	3	1	4	0	3	0	0	0	0
Student 11	0	1	0	2	0	0	1	2	1	2	3	3	0	2	0	0	0	3	0	3
Student 12	2	2	0	2	1	0	0	3	3	2	0	3	1	4	0	4	1	4	4	4
Student 13	2	2	2	2	1	1	3	3	1	3	3	3	0	4	0	4	0	0	0	3
Student 14	0	1	0	0	0	0	0	3	3	3	3	3	0	4	4	0	0	4	3	4
Student 15	2	2	0	2	0	1	4	3	3	3	0	0	0	4	0	0	0	0	0	4
Student 16	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
Student 17	2	2	0	0	0	0	0	0	0	3	0	0	0	4	0	1	0	0	0	0
Student 18	2	2	0	1	0	2	3	3	1	3	3	3	4	4	0	4	0	3	4	4
Student 19	2	2	0	1	0	0	0	2	0	3	0	3	0	1	0	4	0	0	0	0
Student 20	2	2	2	2	1	2	3	3	3	3	3	3	0	4	0	4	1	0	4	4
	33	37	8	21	5	9	24	39	27	49	24	41	12	55	12	33	3	26	25	43
	83%	93%	20%	53%	13%	23%	60%	65%	45%	82%	40%	68%	15%	69%	15%	41%	4%	33%	31%	54%
		40%					65%				22%			59%				18%		43%
	Pre Assessment Data - AO1 Questions					Post Assessment Data - AO1 Questions					Pre Assessment Data - AO2 Questions			Post Assessment Data - AO2 Questions				Pre Assessment Data - AO3 Questions		Post Assessment Data - AO3 Questions
Absolute Difference	24%										37%							26%		
Percentile Difference	60%										169%							146%		

Figure 7. Performance Data on Pre- and Post- Assessment on Ratio.

The overall students' performance on Post Assessments was higher in Ratio topic by 10%. Particularly, students' work samples revealed that the use of bar models and fraction circles seemed to help students to support their solving methods.

Students Surveys on Fractions

Students answered five questions on their preferred representations and methods to answer questions on fractions. More than half of students chose to use sets of objects. Bar models 39% that preferred bar models. Fractions circles was students third choice, while fraction squares and number lines seemed to be the least preferred representations.

The survey revealed that 56% of students found drawings helpful when working with fractions (see Figure 8). An interesting finding was that 30% of students wants to use representations, however they are not sure if they use them properly. Only, 13% found representations not helpful while working with fractions.

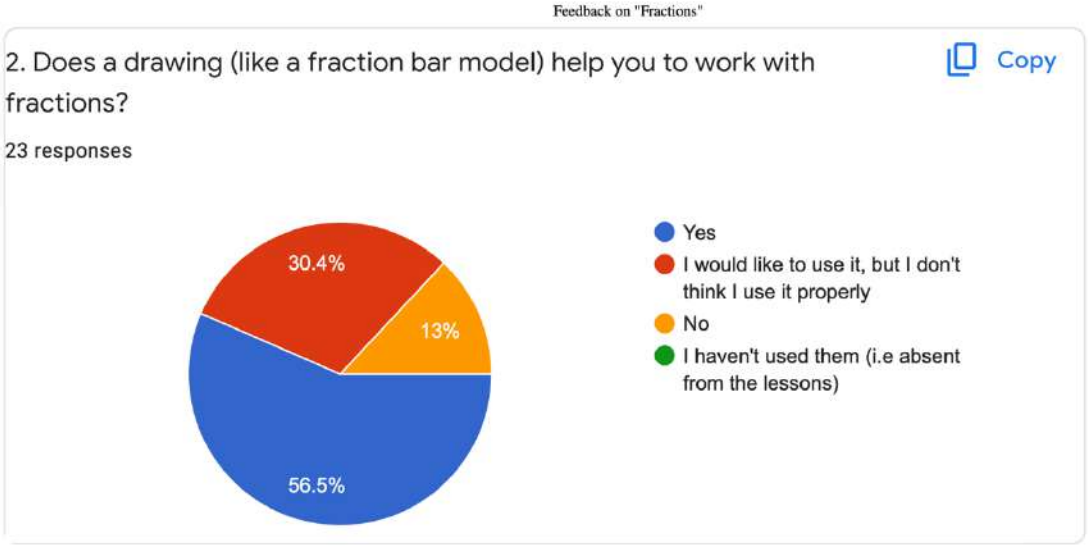


Figure 8. Responses from the Student Survey on Fractions.

Overall, students considered drawing models to be more helpful to answer word problems, fraction/percentage of an amount and reasoning (prove/right or wrong) questions. Students found models to be more useful on reasoning and problem-solving questions compared to "recall knowledge" questions. Also, five out of 23 students did not find drawings helpful when working on fraction questions.

Students were also asked if there was any "trick" method that they have learned in the past and found it helpful. About half of the students (48%) haven't learned an efficient method in the past that found it efficient. About the same percentage of students (44%) said that they have learned a method to multiply and divide fractions. The method was mentioned is the "KFC" (keep, flip, change) method.

Students Surveys on Ratio

Students identified bar models and sets of objects as the most helpful representation while working with ratio (see Figure 9). Only four students out thirty found ratio tables

useful and eight students found abstract methods effective. Also, 67% of students found representations helpful, and 41% wants to use representations but they don't believe they use it properly.

Figure 9. Responses from the Student Survey on Ratio.

Interviews

The assessments revealed a pattern of misconceptions. Overall, students had difficulty to remember a valid method to perform operations with fractions Interviews with eight students were conducted to investigate it further. We interviewed students who did or didn't use representations to answer the questions of the activities.

Samples of students' responses to the interview questions

The student's work in Figure 10 shows that the student didn't attempt to use a visual representation to explain their solutions. They were trying to recall procedural methods they had learned in former school years. Their answer was brief, and provided no explanation of their solving method.

QUESTION 4

Find and fully simplify $\frac{2}{3} \div \frac{2}{9} =$

Students' answer: 1/3

Student explanation: "I divided 2 by 2 and 3 by 9"

Figure 10. Sample work of an unsuccessful attempt to perform a procedural method.

Another group of students seemed to struggle to compare fractions using abstract methods. The interviewees were the students who chose to attempt visual representations. Below is outlined a sample of three students' responses:

Teacher: "I noticed you changed the way you use drawings on Q7, 8 on the pre assessment. You erased the circle and switched to use bar models. Why did you change your mind?"

Student: "I could compare them more easily."

Teacher:" Why did you choose circles to answer question 8?

Student: "I know circles work because I used them last year. I wanted to try the bar way because I haven't used it before to try it how it works."

Cycle 2

Findings on Fraction Activities

In Cycle 2, the project further investigated the student's preferences on their reasoning approaches, as well as the effectiveness of their chosen methods.

Looking through the students' samples on the first five fraction activities (see Figure 11-14), a main finding was that students struggled to draw models representing the concept of "*one whole*". For instance, students added the parts of inequal "wholes" (see Figure 11). Another student struggled with "sets of object" representation, but fraction circles worked efficiently to them (see Figure 12). There was also an attempt to use bar models improperly in fraction notation (see Figure 13). Overall, equi-partitioning fractions was another problematic area that was revealed out of these activities.

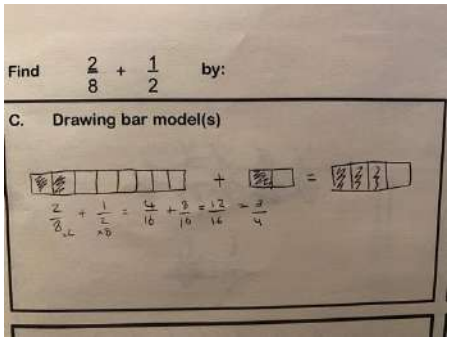


Figure 11. "Wholes" of different sizes.

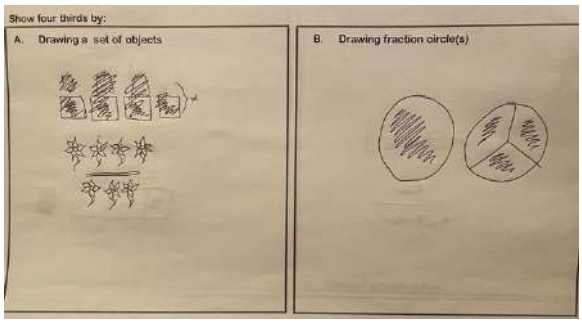


Figure 12. Fraction circles vs. set of objects model

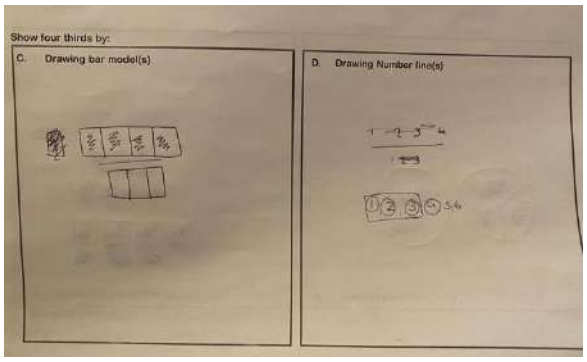


Figure 13. Improper use of Bar models

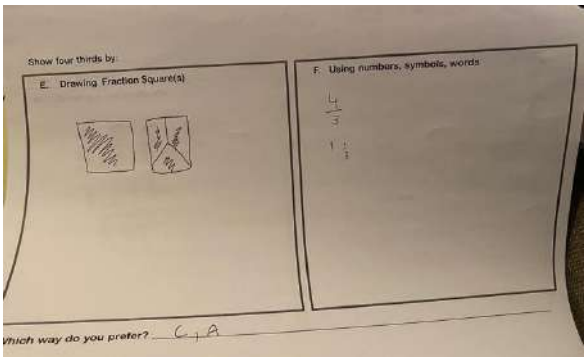


Figure14.Partitioning in inequal parts

Division Question - followed by the Interview

The data for the division activity were organizing by using the coding method. For each student the method of the students was categorized by the pictorial representation they used (i.e., fraction circles), and also indicated on, if the method was successful or not. For the activity on dividing fractions, 15 students chose to use an abstract method. The use of abstract methods had a success rate of 20% (3/15 students). Seven students used simplification which led them to incorrect answers. Three out of fifteen students who chose abstract methods used the "KFC" (*keep- flip-change* method, taught in previous school settings) and only one out of three applied it successfully. The rest five students tried random unsuccessful methods. One student attempted an abstract approach unsuccessfully.

When the same student switched to fraction circles and bar models combined, the student answered the question correctly (see Figure 15).

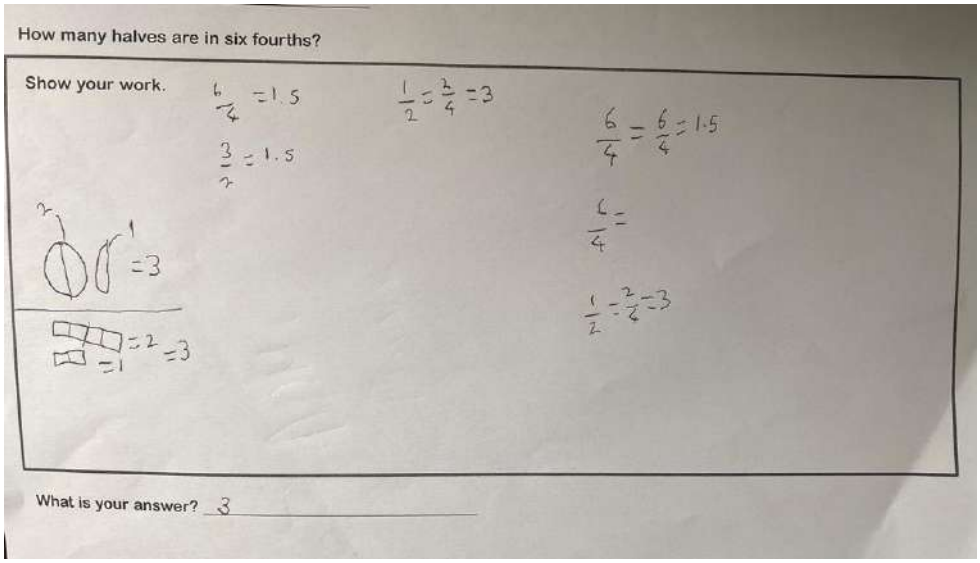


Figure 15. Student switches to pictorial representation.

The rest of the students (10 out of 13) chose to use pictorial methods and had a success rate of 60%. The models that students used were the bar models (80%), and fraction circles (20%). There was no other type of pictorial representations attempted by the students.

Interview Responses

After the completion of the activities, video- based interviews were conducted to investigate further the reason of students' chosen methods and models to represent their answers. The students of the interview group consisted of students who use visual representations both effectively and ineffectively. The interviews aimed to give an insight on student's thoughts on choosing a solving method. Students were asked to identify the model they found helpful, and they were also asked about their prior experiences with representations in mathematics at their former school settings.

Four out of five students said that the model they have used was helpful and all four referred to bar modeling. The fifth student used fraction circles and found it "sometimes helpful".

Regarding their past experiences, students recalled their memories from their maths classes in primary and secondary schools. Most students said that they could not remember the methods that were taught in primary school, and one student recalled fraction circles. In secondary schools, four students recalled abstract methods, and one referred to fraction circles.

Students also responded to an open-ended question on if they remember any methods tricks taught in school. Three students could not remember any, and two students attempted to recall taught methods, such as KFC (Keep, Flip, Change) method and memorized phrases to find a fraction of an amount:

Teacher:

"Any shortcut/trick method that you can remember of?"

Student:

"… KFC for adding subtracting fractions"

The above method has been used to divide fractions. However, the student recalls the name of the strategy and cannot remember in which operation to apply it.

Another student recalls the method to find a fraction of an amount, however the student couldn't memorize the phrase properly:

"…times by the bottom times by the top"

Overall, none the five students could remember or recall effectively a "trick" strategy or to memorize taught phrases to solve questions on fractions.

Fraction Division activity at Sunderland College

The same fraction division activity was given to a group of seven students at Sunderland College. The students have not been exposed to CPA approach. The teacher had presented the students bar modeling when taught ratio. It is important to mention that department's Scheme of Learning allows for three lessons of ratio to be delivered throughout the academic year. The teacher found that bar modeling helped students to work independently and to solve problems on ratio effectively. Students used mostly abstract methods and one out of seven students answered the question correctly. The two students that chose to use pictorial approaches were drawings of the actual objects (water bottles) and were used inaccurately (see Figure 16). Also, the fraction notation shows that student does not understand the meaning of the fraction representation in numbers and words.

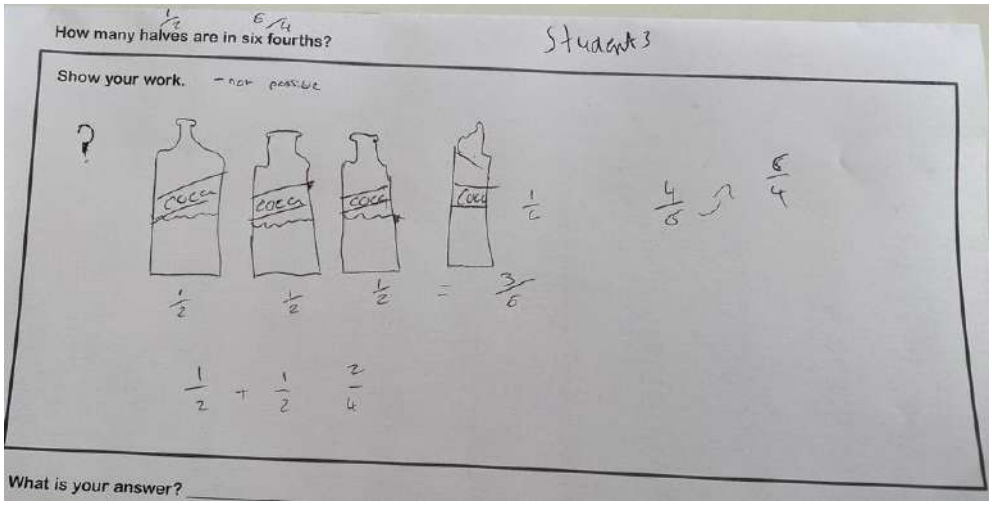


Figure 16. Student draws the actual object to work out the question.

The students were asked the same interview question to those that were used at Gateshead College. The students' answers were recorded using note taking. Overall, students' responses revealed misconceptions about the whole. Also, students did not apply the bar modeling or any other type of visual representation effectively. Samples of the students' responses are listed below:

-Question: "What is considered as a "whole"?"

-Student 1: "A whole block I guess"

-Student 2: "Whatever matches the denominator the numerator counts as a whole"

-Question: “Any tricks/methods that you have used in the past that they have worked with you?”

-Student 3: “Not really just divided the bottom number by the top number”

-Student 4: “…I just... nothing comes to mind”

-Student 5: “I am not sure if I haven’t seen this type of question before, …I don’t like to use words in maths.”

Conclusions and Recommendations

Conclusions

- Based on the data, students’ performance on reasoning and problem-solving questions showed a significantly higher improvement compared to recall/use knowledge questions on both fractions and ratio post-assessments. Students seemed to struggle more on AO1/recall questions, and they were more reluctant on using concrete and/or pictorial representations to answer them.
- Assessments as well as open response questions during the lessons revealed misconceptions on basic concepts such as representing a whole. CPA approach offered opportunities to identify significant misconceptions that, furthermore, impedes students from developing reasoning.
- Students tend to repeat methods taught in secondary school, that cannot remember and apply properly. They described the solving strategies from secondary schools, more as abstract procedural methods. Students found themselves trapped in shortcuts and math tricks that longtermly they tend to forget. This learning experience has an impact on students’ development of their reasoning skills. Students need teaching approaches that will enhance deeper understanding, such as working with concrete and pictorial approaches prior to moving to more abstract methods.
- Bar modelling appeared to be the most efficient model that students chose to answer questions on fractions and ratio. Bar models have been applied more efficiently compared to other models. Most of the students chose to use the bar models, and a small number of students used the fraction circles.
- From the student surveys in both Gateshead and Sunderland college, we saw that a significant number of students showed an interest to use pictorial representations to help them showing their work, however they felt unsure on if they use them properly. Students may need more time to practice those strategies.
- The scheme of work in Further Education leaves teachers with the challenge to teach a long list of topics in limited time. Students need more time to explore and use strategies for deeper understanding of the topics.

Recommendations

- Students have difficulty to recall and apply learned methods from the primary and/or secondary school years, such as shortcuts and tricks. For that reason, teacher may choose to apply methods that encourage deep understanding of the topics. However, valuing students’ methods that are already effective, is as important as to provide students with alternative strategies as needed.

- Identifying misconceptions can strengthen the prior knowledge foundation that is needed before introducing a topic. It will also better inform our teaching and can help students use the methods they choose more effectively. For example, the use of the diagnostic questions, and/or opportunities for classroom discussion can reveal possible misconceptions. Students need to be encouraged to explain their answers and avoid the “right or wrong responses”.
- Students were very open to try new methods and were involved in all three levels of the CPA model. There was no hesitation to use concrete maths manipulatives, to facilitate the transition from one stage to another (e.g., from concrete to pictorial).
- Although, the data thought showed us promising outcomes from the use of CPA approach and its positive impact on reasoning skills development, conclusions and recommendations should be considered with caution due to the small student sample size we used in this project.
- Time restrictions on teaching and preparing resources appeared as the two main factors that impede proper application of CPA model. We may want to reconsider the sequence of our scheme of work, based on the and how the topics are connected.

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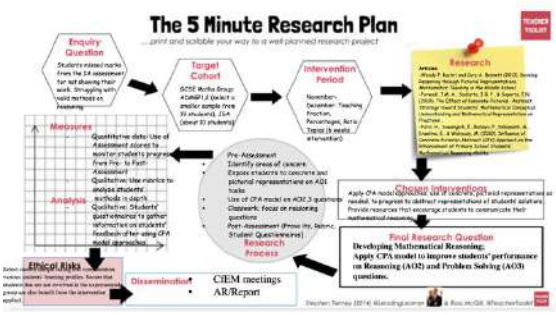
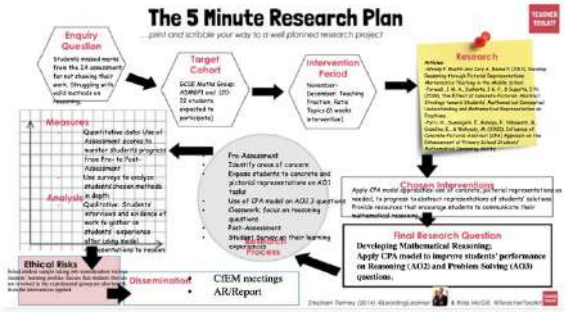
Appendices

Appendix 1. Five Minute Research plan (https://www.teachertoolkit.co.uk/product/the-5-minute-research-plan/





Appendix 4

Pre -Assessment **Fractions** **GCSE Maths**

Name: _____

Date: _____

Vocational Area: _____

Assessment Objective	Question	Topic	Not attempted	Partially answered	Answered correctly	Marks	Score
AO1	1	Equivalent fraction				/1	
AO1	2	Multiply, Simplify fractions				/2	
AO1	3	Convert a mixed number to an improper fraction				/1	
AO1	4	Divide, Simplify fractions				/2	
AO2	5	Identify error on adding fractions				/2	
AO2	6	Compare, order FDP				/2	
AO2	7	Reason on "right or wrong" question (compare fractions)				/2	
AO2	8	Justify conclusion on which fraction is closer to a half				/2	
AO3	9	Problem solving on fractions of a quantity				/3	
AO3	10	Contextualised problem solving				/3	
		Total					

QUESTION 1

Find an equivalent fraction to $\frac{12}{36} = \_$

QUESTION 2

Find and fully simplify $\frac{6}{8} \times \frac{4}{3} =$

QUESTION 3

Write the mixed number as an improper fraction $3\frac{2}{5} = \_$

QUESTION 4

Find and fully simplify $\frac{2}{3} \div \frac{2}{9} =$

QUESTION 5

Explain in the box below why the following is incorrect: $\frac{3}{4} + \frac{2}{5} = \frac{5}{9}$

Answer box

QUESTION 6

Place $\frac{3}{2}$, 120%, 0.900 in ascending order (from smallest to largest).

Show your work in the answer box.

Answer box

QUESTION 7

Taylor says that $\frac{3}{4}$ is greater than $\frac{3}{5}$

Is Taylor right? Show your explanation below.

Answer box

QUESTION 8

Which fraction is **closer** to $\frac{1}{2}$

$\frac{2}{6}$ or $\frac{5}{8}$

Explain your answer below. Show your work.

Answer box

QUESTION 9

Two friends buy a holiday package. One pays £800 and the remaining $\frac{2}{3}$ of the total cost is paid by the other. How much does the holiday package cost in total?

Show all the steps of your work below.

Answer box

QUESTION 10

The area of the floor of an exhibition centre is $140m^2$. The $\frac{1}{7}$ of the floor is covered with black tiles the rest is covered equally with white and red tiles. What is the area covered with white tiles?

Show all the steps of your work below.

Answer box

Pre -Assessment **Ratio** **GCSE Maths**

Name: _____

Date: _____

Vocational Area: _____

Assessment Objective	Question	Topic	Not attempted	Marks
AO1	1	Write an integer ratio in its simplest form		/2
AO1	2	Convert and write a ratio in its simplest form		/2
AO1	3	Justify if the ratio of the scale is right or wrong		/2
AO1	4	Share an amount in a given ratio		/3
AO1	5	Use ratio to identify unknown parts		/3
AO2	6	Identify and justify errors on writing a ratio in the form 1 : n		/3
AO2	7	Work out missing ratio quantities το calculate the total		/4
AO2	8	Solve comparison ratio word problem		/4
AO3	9	Solve best buy problem		/4
AO3	10	Solve recipe problem		/4
		Total		

Pre -Assessment Ratio GCSE Maths

QUESTION 1

Write the ratio 8 :12 in its simplest form**.**

Show your working.

QUESTION 2

Write the ratio 95p to £3.00 in its simplest form.

Show your working.

QUESTION 3

"A map has a scale of 1cm represents 50m. Its equivalent ratio is 1:50."

Is the above statement correct?

Explain your answer.

QUESTION 4

Share £360 in the ratio 4 : 5.

Show your working.

QUESTION 5

Dylan and Shawn share 70 sweets in the ratio 2:3. How much does Shawn get?

Show your working

QUESTION 6

In nursery there are 5 adults for every 15 children.
Ben says that the ratio of adults to children in the form 1 : n is expressed as 1 : 11
Explain why his answer is incorrect.

Show your working.

QUESTION 7

Luke and Paula share the expenses of a trip in the ratio of 4: 5. Paula pays £450. How much is the cost of the trip all together? Show your working.

Show your working.

Pre -Assessment Ratio GCSE Maths

QUESTION 8

Josh, James and John share biscuits in ratio 1 : 2 : 4. Josh has 9 less biscuits than John.
How many biscuits does James have?

Show your working.

QUESTION 9

A supermarket sells packets of washing powder in 3 sizes.

Size	Price	Number of washes
Large	£11.00	45
Medium	£5.59	23
Small	£3.50	10

Which is the best value for money?

Show your working.

QUESTION 10

Dan is making ice cream. He is making the recipe below.

serves 4
300ml double cream
320ml milk
120g caster sugar
1 vanilla pod
4 egg yolks

How much of each ingredient would Dan need to make enough for 6 people?

Show your working.

Appendix 2. Pre/Post assessment Fractions and Ratio

Name: _____

Show four thirds by:

A. Drawing a set of objects

B. Drawing fraction circle(s)

Show four thirds by:

C. Drawing bar model(s)

D. Drawing Number line(s)

Show four thirds by:

E. Drawing Fraction Square(s)	F. Using numbers, symbols, words

Which way do you prefer? _____

Name: _____

Show an equivalent fraction to $\frac{2}{6}$ by:

A. Drawing a set of objects	B. Drawing fraction circle(s)

Show an equivalent fraction to $\frac{2}{6}$ by:

C. Drawing bar model(s)

D. Drawing Number line(s)

Show an equivalent fraction to $\frac{2}{6}$ by:

E. Drawing Fraction Square(s)

F. Using numbers, symbols, words

Which way do you prefer? _____ _____

Name: _____

Find $\frac{1}{2} \times \frac{1}{3}$ **by:**

What is your answer? _____

Name: _____

Find $\frac{3}{2} \times 12$ **by:**

A. Drawing a set of objects

B. Drawing fraction circle(s)

Find $\frac{3}{2}$ x 12 by:

C. Drawing bar model(s)	D. Drawing Number line(s)
E. Drawing Fraction Square(s)	F. Using numbers, symbols, words

Which way do you prefer? _____

Appendix 3

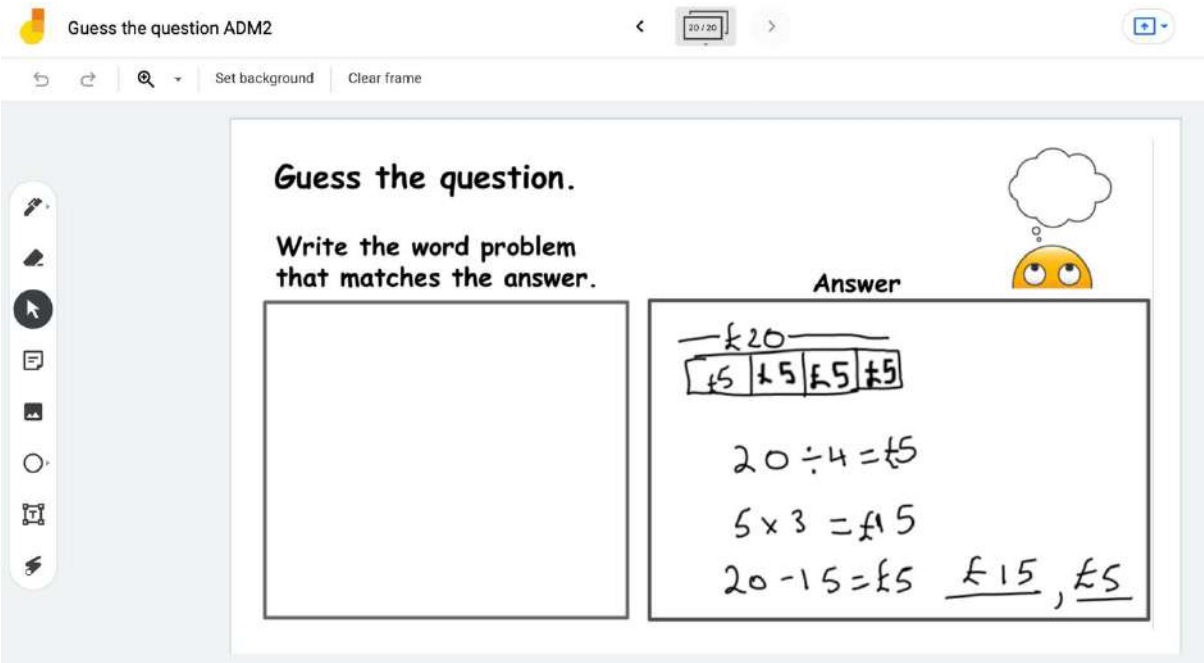
Name: _____

How many halves are in six fourths?

Show your work.

What is your answer? _____

Appendix 5



Appendix 6

Feedback on "Fractions"

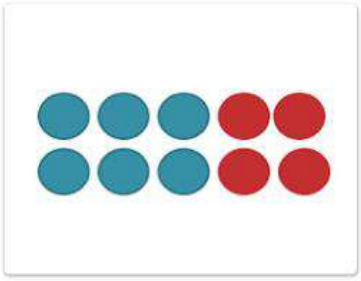
Complete the questionnaire on what has been helpful to you to better understand the topic.

*Required

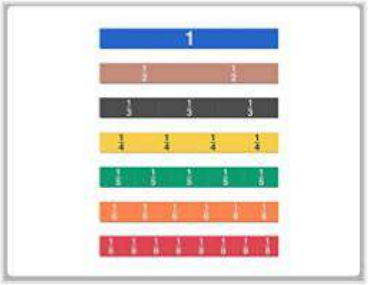
1. Email *

2. 1. Which models did you find more helpful to represent fractions? Check all that apply. *

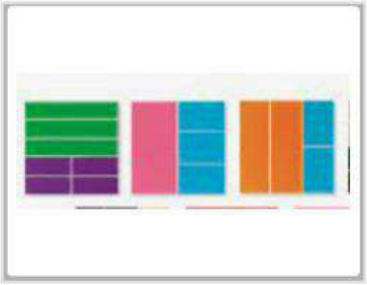
Tick all that apply.



☐ Sets of Objects

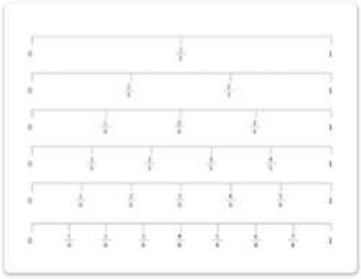


☐ Fraction Bar Models



☐ Fraction Squares

☐ Cicle Fractions



☐ Number Lines

3. 2. Does a drawing (like a fraction bar model) help you to work with fractions? *

Mark only one oval.

- ◯ Yes
- ◯ I would like to use it, but I don't think I use it properly
- ◯ No
- ◯ I haven't used them (i.e absent from the lessons)

4. 3. When is a drawing useful to you? Check all that apply. *

Tick all that apply.

- [ ] Add and Subtract fractions
- [ ] Multiply and Divide Fractions
- [ ] Equivalent Fractions
- [ ] Fraction/Percentage of an Amount
- [ ] Questions that they ask me to prove if something is right/wrong
- [ ] Word problems to organise my thinking/solution
- [ ] Drawing models is not useful to me.

5. 4. Do you use a "trick"/method, other than drawings, that you find more helpful on the following topics? Check all that apply. *

Tick all that apply.

- [ ] Add and Subtract Fractions
- [ ] Multiply and Divide Fractions
- [ ] Equivalent Fractions
- [ ] Fraction/Percentage of an Amount
- [ ] Word Problems
- [ ] No method learned in previous years helped me with the above

6. 5. If you checked any of the first five options on the previous question, name the trick/method you found helpful.

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Google Forms

Feedback on "Ratio-Proportion"

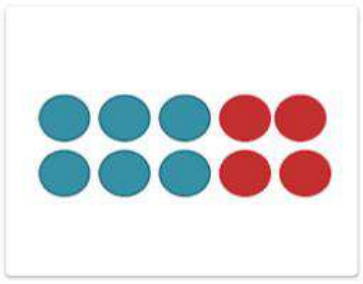
Complete the questionnaire on what has been helpful to you to better understand the topic.

*Required

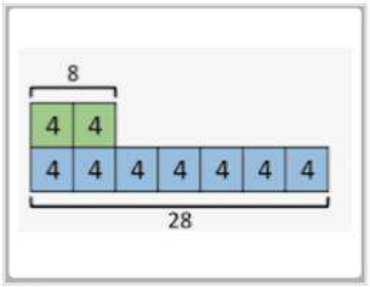
1. Email *

2. 1. Which way did you find most helpful to work on ratio questions? (select all that apply) *

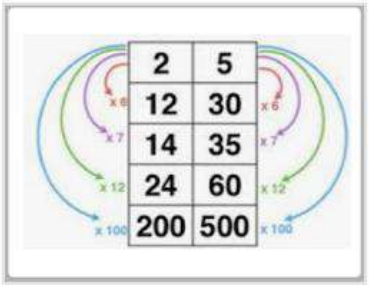
Tick all that apply.



☐ Sets of Objects



☐ Bar Models



- [ ] Ratio Tables
- [ ] Only Words, Numbers, Symbols

3. 2. Does a drawing help you to work with ratio questions? *

Mark only one oval.

- ◯ Yes
- ◯ I would like to use it, but I don't think I use it properly
- ◯ No
- ◯ I haven't used them (i.e absent from the lessons)

4. 3. Check all the cases below, for which you would use a drawing to show your work. *

Tick all that apply.

- [ ] Represent/Show a ratio
- [ ] Simplify ratio
- [ ] Word Problems with Sharing
- [ ] Recipe Word Problems
- [ ] Best Buy Problems
- [ ] Drawing models is not useful to me.

5. Would you use a drawing to show your work at the GCSE exam questions? *

Mark only one oval.

- ◯ Yes, I would use a drawing along with words, numbers and symbols.
- ◯ No, I don't think that drawings is an appropriate way to answer exam questions.

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Google Forms

Appendix 7

Name: _____

Show an equivalent fraction to $\frac{2}{6}$ by:

A. Drawing a set of objects	**B. Drawing fraction circle(s)**

Show an equivalent fraction to $\frac{2}{6}$ by:

C. Drawing bar model(s)	**D. Drawing Number line(s)**

Show an equivalent fraction to $\frac{2}{6}$ by:

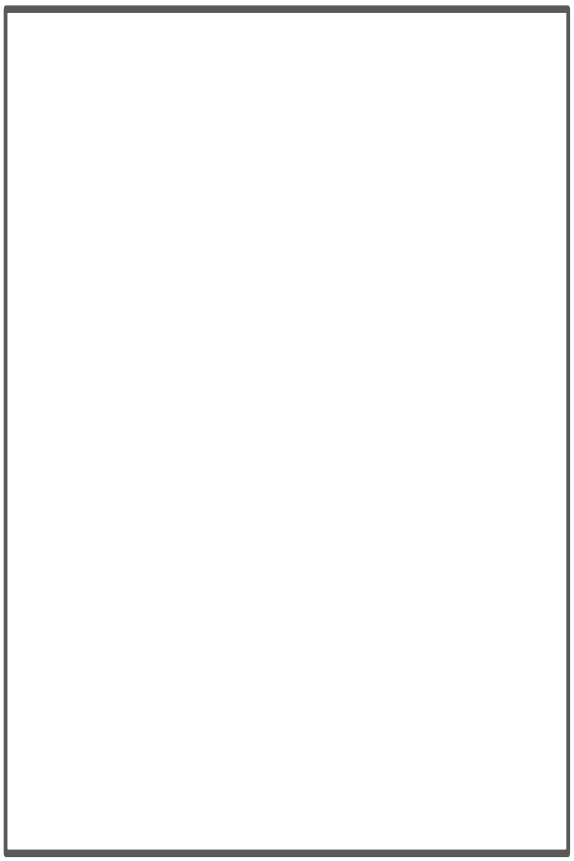
E. Drawing Fraction Square(s)	F. Using numbers, symbols, words

Which way do you prefer? _____

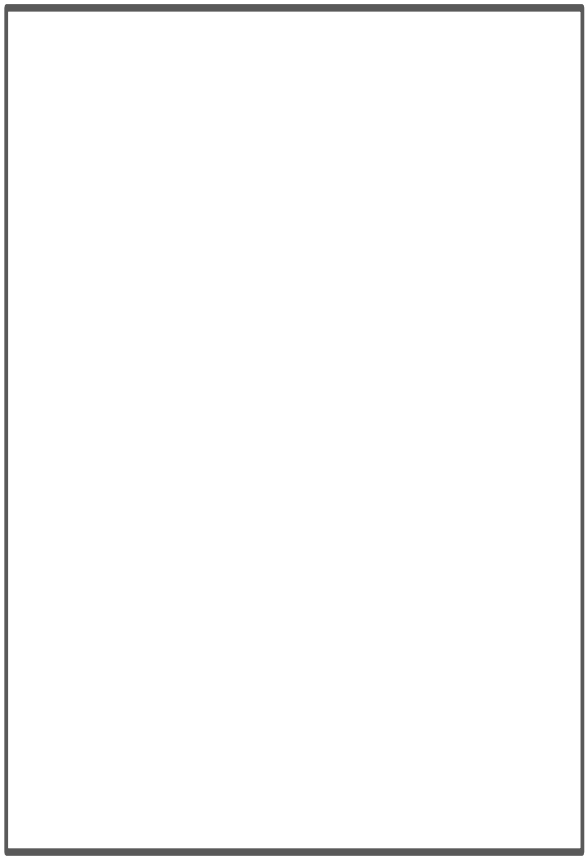
Name: _____

Find $\frac{2}{8} + \frac{1}{2}$ by:

A. Drawing a set of objects



B. Drawing fraction circle(s)



Find $\frac{2}{8} + \frac{1}{2}$ by:

C. Drawing bar model(s)

D. Drawing Number line(s)

E. Drawing Fraction Square(s)

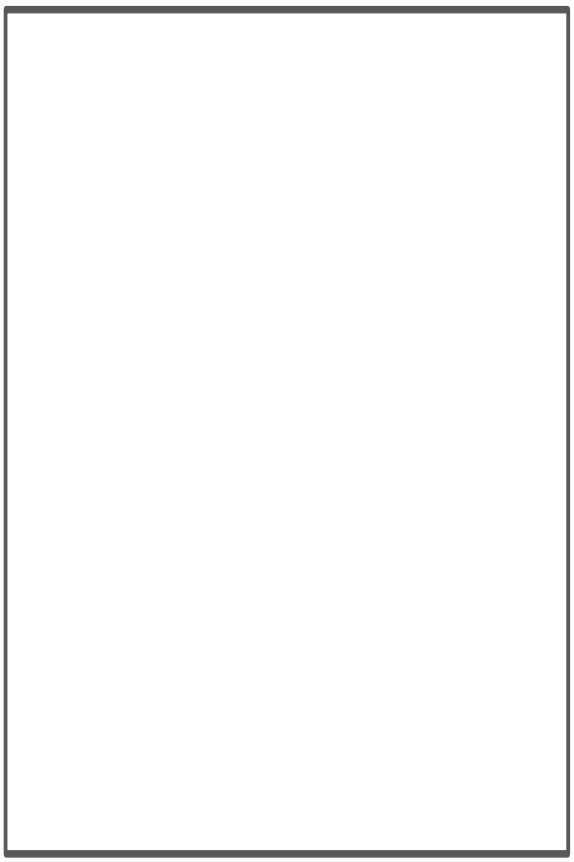
F. Using numbers, symbols, words

Which way do you prefer? _____

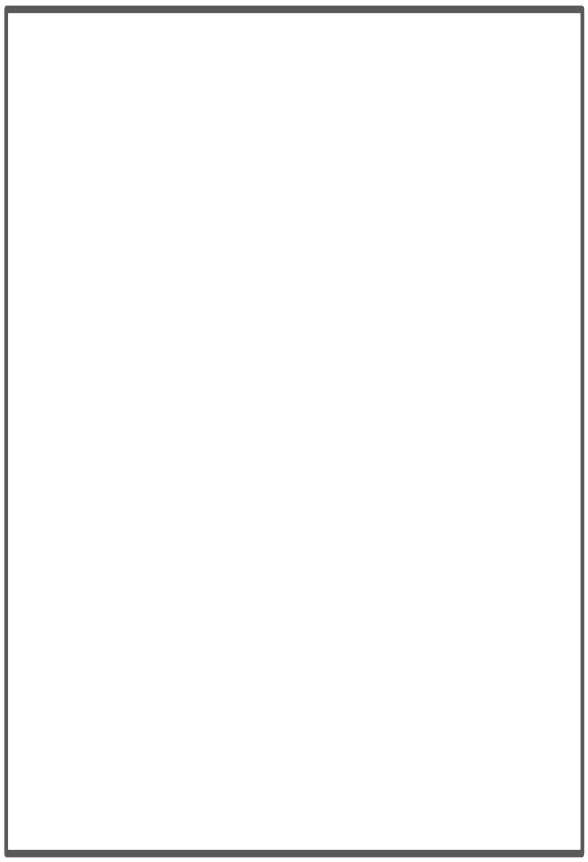
Name: _____

Find $\frac{3}{2} \times 12$ by:

A. Drawing a set of objects



B. Drawing fraction circle(s)



Find $\frac{3}{2} \times 12$ by:

C. Drawing bar model(s)

D. Drawing Number line(s)

E. Drawing Fraction Square(s)

F. Using numbers, symbols, words

Which way do you prefer? _____

Name: _____

Show four thirds by:

A. Drawing a set of objects

B. Drawing fraction circle(s)

Show four thirds by:

C. Drawing bar model(s)

D. Drawing Number line(s)

Show four thirds by:

E. Drawing Fraction Square(s)	F. Using numbers, symbols, words

Which way do you prefer? _____

Name: _____

Find $\frac{1}{2} \times \frac{1}{3}$ **by:**

What is your answer? _____

Name: _____

Find $\frac{3}{2}$ x 12 by:

A. Drawing a set of objects

B. Drawing fraction circle(s)

Find $\frac{3}{2}$ x 12 by:

C. Drawing bar model(s)

D. Drawing Number line(s)

E. Drawing Fraction Square(s)

F. Using numbers, symbols, words

Which way do you prefer? _____

Name: _____

How many halves are in six fourths?

Show your work.

What is your answer? _____