CENTRES FOR EXCELLENCE IN MATHS

LSC
leyton sixth form college
SUCCESS AT A CARING COLLEGE

Using double number lines and bar modelling to teach the GCSE maths curriculum based on the Mastery approach.

Misbah Abbas, Jane Barnett, Jayon Charles, Byron Sheffield

OUR PARTNERS

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Opening Notes

In this project we collaboratively planned a series of lessons for GCSE resit students who had previously achieved grade 1 or 2 using double number lines and bar models as part of a Mastery approach to teaching.

We chose these diagrams because they can be adapted to model a variety of different mathematical topics at GCSE Maths which students have traditionally struggled with and tend to ignore in exam papers such as proportion, ratio, problem solving and solving equations.

Acknowledgements

Thanks go to Cath Gladding for the Online research training, Joss Kang for technical support for the online sessions, and the University of Nottingham for the use of some of their trial materials.

About CfEM

Centres for Excellence in Maths (CfEM) is a five-year national improvement programme aimed at delivering sustained improvements in maths outcomes for 16– 19-year-olds, up to Level 2, in post-16 settings.

Funded by the Department for Education and delivered by the Education and Training Foundation, the programme is exploring what works for teachers and students, embedding related CPD and good practice, and building networks of maths professionals in colleges.

Summary

Students with very low prior attainment lacked strategies to attempt much of the GCSE Maths exam. We focused on using bar models and double number line diagrams as part of a mastery approach to improve conceptual understanding and retention in key topic areas such as Ratio, Proportion, Speed and solving Equations.

We adapted our scheme of work to introduce the diagrams earlier in the year. 4 teachers working with 7 classes (in total 125 students) engaged in a shared planning and lesson study process. Students' attitudes and attainment were measured using an online survey after each research lesson. We also examined their performance on key questions in whole-cohort formal assessments.

We found that students need support to draw the diagram well enough for it to be useful. Students who did not previously have a method for solving these problems were more likely to adopt the diagram method we used for the mastery approach. Students who did have an alternative method were more reluctant to try the diagram and some found it confusing - it’s important to connect the diagram to other methods so students see how it can be applied over various mathematical topics.

Some students did go on and use bar models and double number lines in assessments which were GCSE exam papers. This gave them a way to access questions they might otherwise have not attempted. In surveys after the intervention lessons, students said that they liked and valued the diagram methods, even if they did not go on to use them on their own initiative in the assessment.

Our key finding was that students who do not have a method or who struggle with traditional methods are more likely to attempt to solve a problem if they have learned how to picture the information on a diagram. We also found that some students who had learned how to use the diagrams were able to transfer it to different contexts or into the assessment. We developed our pedagogy for effective teaching using these diagrams to increase the impact on understanding and likelihood that students would transfer the diagram to a new context.

Having trialled this with students entering with grade 0-2 we are now confident to widen the approach to more topics and all GCSE resit students and to support other teachers to be confident in delivering this approach.

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Background

Our college and cohort

"Working within a safe, welcoming and stimulating environment, which embraces diversity and promotes respect, we help students fulfil their academic potential and become thinking, questioning and caring members of society."

Leyton Sixth Form College has about 2000 students, mostly aged 16-19 and studying full time at level 3. Around 60% of students are doing A-levels and 40% are on vocational programmes such as BTEC. We also offer BTEC and ESOL courses at Level 1 and 2 to enable students to access further learning through progression at the college. Around 600 plus students go on to university each year from both A Level and Vocational courses. Nearly a hundred students gained places at Russell Group universities last year.

"*Waltham Forest is currently ranked 82nd most deprived borough nationally according to the 2019 Index of Multiple Deprivation (an improvement from 35th in the 2015 edition, and 15th most deprived in the 2010 edition)."*

https://www.walthamforest.gov.uk/content/statistics-about-borough

We have been a member of the Centres for Excellence in Mathematics (CfEM) since October 2018. We participated in the CfEM research project National trials for Mastery with the University of Nottingham in 2019-2020.

Our learners and our goals for GCSE Maths

Since it became mandatory for students who had not achieved a "pass" (C or 4) at GCSE to resit, the GCSE resit programme has grown from around 200 students to around 600 students. We offer GCSE Maths to all students who have not yet achieved a grade 4. We split this cohort into two courses, one for students who have a grade 3 and are working towards a grade 4, and one for students who have less than grade 3 with the goal of achieving a grade 3 and progressing to the next level alongside the other courses they are doing the following year. The value added on these courses is excellent, and overall students on the 3-4 level do better than the national benchmark for success in GCSE resit Maths, but we are ambitious for more of our students to pass GCSE Maths before they leave college. (You can see more about students' grade progression through their career at LSC on the Finishers table on the next page.)

Here's how the students on different courses do compared to the cohort doing GCSE resit with the exam board (Pearson Edexcel). It's important to bear in mind that many FE colleges prefer to offer Functional Skills qualifications to students who have less than grade 3 at GCSE Maths.

Students achieving grade 4+ in GCSE Maths	2018	2019	2020
All FE students (Pearson)	19.40%	18.20%	unavailable
All LSC students	19.85%	16.90%	25.40%
3-4 course at LSC	30.42%	29.80%	42.55%

0-3 course at LSC (NB target grade is 3 not 4)	1.99%	1.52%	1.61%

You will see there is a significant increase in 2020 – although we have taken steps to introduce Mastery teaching in 2020 there was also a pandemic and students received CAGs rather than doing an exam, so it is not strictly comparable.

This table shows entry grades and final grades in GCSE Maths for students leaving LSC in 2019. This analysis represents the fact that students may take GCSE Maths for 2 or 3 years.

FINISHERS SUMMER 2019

Entry Grade	No.of students at this grade	Numbers achieving grade indicated							Percentage of students who dropped back at least one grade	percentage of students who stayed at the same grade level	Percentage of students who progressed by at least one grade
		X/U	1/FG	2/E	3/D	4/C	5	B/A			
3/D	186	3	1	21	74	79	1	0	13.44%	39.78%	43.01%
2/E	93	4	9	38	23	8	0	0	13.98%	40.86%	33.33%
1/FG	43	8	18	12	1	0	0	0	18.60%	41.86%	30.23%
0/U	10	2	5	0	2	0	0	0	-	20.00%	70.00%
X/NONE	38	7	1	2	0	4	0	1	-	18.42%	21.05%
	370	24	34	73	100	91	1	1	Progress Score: 0.30421		

This similar table shows entry grades and final grades in GCSE Maths for students leaving LSC in 2020, although this does include some grades from CAGs due to the pandemic so it may not be directly comparable.

FINISHERS SUMMER 2020

Entry Grade	No.of students at this grade	Numbers achieving grade indicated							Percentage of students who dropped back at least one grade	percentage of students who stayed at the same grade level	Percentage of students who progressed by at least one grade
		X/U	1/FG	2/E	3/D	4/C	5	6 | 7			
3/D	153	0	0	0	66	80	4	3	0.00%	43.14%	56.86%
2/E	90	0	0	41	37	9	3	0	0.00%	45.56%	54.44%
1/FG	40	0	15	16	8	1	0	0	0.00%	37.50%	62.50%
0/U	8	1	2	3	0	1	0	1	-	12.50%	87.50%
X/NONE	29	8	3	4	8	6	0	0	-	27.59%	72.41%
	320	9	20	64	119	97	7	4	Progress Score: 0.74914		

Our project: Key diagrams for conceptual understanding

We chose to focus on developing conceptual understanding of some core concepts of GCSE Maths - Ratio, proportion, speed and solving equations - with students on the lowest level of GCSE Maths. These students have experienced repeated failure over 5 years of secondary school and need to overcome significant cognitive and emotional barriers to be successful in the 2-3 years they are at college. We have selected the Pictorial aspect of the Concrete-Pictorial-Abstract pathways and chose two diagrams - bar models and double number lines - which can be adapted to multiple different areas of the GCSE curriculum to provide best value from the time taken to learn how to draw and use the diagrams. Once we've established how and that this works, we intend to expand the approach to other topics and the students on the GCSE grade 3-4 level.

Literature Review

The experience of students learning mathematics in this country involves an accelerating one-way march through the concrete and pictorial to the abstract. Particularly from the early years of Key Stage 3, students are assumed to have the understanding to develop familiar and new topics through the medium of signs, symbols, and algorithms. Although this is true for some students it becomes a real issue for learners who cannot make the connections to their previous understanding and struggle to hold the increasing amount of knowledge required in their working memory. This leads to misconceptions that become entrenched and a growing lack of confidence amongst some learners. Post-16 GCSE resit students often have incorrect or inflexible models for solving mathematical questions or even no model leading to random applications of partly remembered rules.

There is however a growing movement within many primary and some secondary schools based on research and teaching approaches from places abroad such as Singapore that recognises the importance of the Concrete-Pictorial-Abstract continuum and the need for students to move backwards and forwards within this as they develop their understanding. The role of representations as a key part of this approach has been developing for some time and our action research group was interested to see what the implications of using them would be within a two-year GCSE course, where time is tight, students come with very different experiences and mathematical approaches and staff have limited experience of using them.

We are hopeful that by offering our students two new, highly versatile representations (bar models and double number lines), they will be able to model and thereby make progress to solve a greater variety of problems, in particular worded questions on the GCSE exam. This is because it has been seen that...

"*When students gain access to mathematical representations and the ideas they express and when they can create representations to capture mathematical concepts or relationships, they acquire a set of tools that significantly expand their capacity to model and interpret physical, social, and mathematical phenomena." NCTM* (2000)

The use of the representation is not only seen as supporting the student to solve a problem but also and importantly in the development of their understanding of the mathematics. This plays a central role within the Realistic Mathematics Education framework that is used widely in Holland and beyond.

"Students pass through different levels of understanding on which mathematizing can take place:

From devising informal context-connected solutions to reaching some level of schematization, and finally having insight into the general principles behind problem and being able to see the overall picture."

"In the beginning of a particular learning process a model is constituted in very close connection to the problem situation at hand, and that later on, the context specific model is generalized over situations and becomes then a model that can be used to organize related and new problem situations and to reason mathematically". Heuvel-Panhuizen (2003)

Importantly the representations are not there to prescribe the methods students used to solve the problems but encourage them to use a range of approaches. "It is not the models in themselves that make the growth in mathematical understanding possible, but the students' modelling activities" Heuvel-Panhuizen (2003).

Research with 12-year-olds by Pantziara et al (2004) compared the result of two tests. In one the problems were presented without diagrams and in the other the problems were presented with diagrams. Like us the researchers chose to focus on a limited number of abstract standard diagrams so that they could be transferred into many different contexts. The difference in results was not statistically significant, but they note that different students were successful on the two different tests. Working with students aged 16+ on GCSE resit, we can be sure that they were not successful in school, so trying a new approach may give them a chance to be in the successful group.

However, attempting such new approaches within this short course brings its own issues given the potential years of perceived failure and entrenched misconceptions. Research into adopting a mastery approach in schools by Jerrim and Vignoles (2016) highlighted the issues we face. After several randomised cluster trials, they found that the impact of the mastery approach was much greater in the primary schools than in the secondary schools. The researchers also found that schools that were graded as good or outstanding had the greater improvements in both primary and secondary.

It was found that the mastery approach has an impact on learning, but it needs to be implemented in the early years of a child's education so that students have a chance of using the approach to develop confidence and they become accustomed to the techniques and methods. This is shown in the greater improvement in the primary schools.

So, given the less-than-ideal situation of trying to implement these approaches at such a late stage of the student's mathematical journey, will our chosen representations help and how can we ensure that they have as much impact as possible? There is very little research on this at FE level, but lessons can be learned from lower age groups.

A study in 2020 by Low, Shahrill and Zakir looked at the use of bar models to support learning with a variety of fraction calculations as well as the use of an algorithmic picture (Butterfly) method to add and subtract fractions. They found that bar models had a positive effect on students' ability to both perform the calculations and as a basis to explain their method and why it worked. The Butterfly method improved student's ability to add fractions, but students struggled with subtracting or dealing with improper fractions. Those students who could use both methods found it useful to be able to check their answers.

Another in 2021 by Said and Tengah identified that students in their context underachieve in Maths exams compared to their other subjects. They planned an intervention of 3 x 1-hour lessons on ratio using bar models. By comparing pre and post test scores for the classes who had the intervention, they found all students had made progress but the students in the lower ability group had made the *most* progress. This is relevant to our study because we have a similar context in which students tend to underachieve in GCSE Maths compared to other GCSEs and we are focused on working with lower ability students who struggle to understand what to do in worded problems.

Double number lines (DNL) are a representation of proportionality that underpins a wide range of topics within the Maths GCSE that we currently tend to teach separately. There are several other proportional representations including the ratio table that currently seems more regularly used by teachers. However, our experience of using DNLs within the University of Nottingham national mastery trials that two of our teachers took part in previously led us to want to see how focussing on this approach would help students.

Research by Küchemann, Hodgen and Brown in 2011 showed that DNLs can be used to emphasise scaling as a better approach than additive strategies when teaching multiplicative reasoning. The researchers recognised that a multiplicative reasoning is a long-term process and so DNL's should be introduced at the earlier stages of education and, therefore, introducing DNL's to 16–19-year-olds inevitably comes with its challenges. Applying the technique to various topics such as currency conversions, units of measurement and ratio and proportion throughout the year helped our students to adopt the approach more readily, however, more time is needed to get students to use the approach in an exam rather than in the lessons. Many GCSE students do not have a concrete method for multiplicative reasoning and revert to repeated addition when solving problems. In conclusion, the researchers recognised that spending time developing an understanding of DNLs will help students to make the cognitive shift in their understanding of multiplication. If these models are used regularly in our teaching, we can hopefully convince students to use them confidently and use them successfully.

Similarly, Misailidou (2007) examined the use of a prepared Double Number Line diagram introduced partway through a discussion in a small group of 11-year-olds about their own methods of solving a direct proportion problem. As a result, the focus of students changed from having a common misconception (adding the same amount to both quantities) to a correct multiplicative strategy. Her justification for this new approach was reliant on the use of the DNL. As this shows that a DNL diagram can help to change students' misconceptions around scaling it could prove a very useful tool with resit students.

As a tool for developing students' thinking more fundamentally than their ability to solve problems, Archer, Morgan & Swanson (2021) observed a collaboratively planned lesson in Japan where DNL diagrams were used to support students' explanations of their reasoning in a Best Buy problem. At first whole numbers were used to reduce the demand of the division / multiplication and allow students to focus on the diagrams, process and reasoning. Then an additional product was added that required division by decimals. This additional challenge meant many students were less certain of their strategy and could no longer clearly explain what their answer meant. The researchers note that in the collaborative meeting afterwards the teachers agreed it was essential that students attempt to explain why they chose their method and what it meant – that this was vital for them to remember it and understand it deeply rather than learn an algorithm which they might then misapply through not really understanding what it meant in context.

Finally, we were interested to know what type of training and support would be useful for teachers to be able to implement these approaches in a way that instilled confidence with students. Research by Orril and Brown (2012) into CPD sessions for teacher on how to use DNLs to support learning highlighted that "the importance of MKT (Mathematical knowledge for teaching) as distinct from just mathematical knowledge" and that "there has been a lot of research on student learning but far less focussed on teachers understanding". They concluded that teachers need CPD to help them understand how DNLs can be used for reasoning about proportional relationships (not to make calculations easier) and how to make connections between different areas of mathematics.

In conclusion, there is good evidence that a Mastery approach in general and the thoughtful use of these specific diagrams can improve students learning at an earlier stage in their academic career so there is reason to believe that it is worth experimenting with these techniques with students doing GCSE resit, especially if they have not tried these approaches before.

Methods

We conducted 4 cycles of collaborative planning, teaching, gathering evidence and reflection. We reviewed the literature on teaching the topic using diagrams and found mastery resources. We then collaboratively planned the lessons based on our research. We practiced and made sure we had mastered the techniques before teaching the lesson to our classes - it is vital that we are confident in the use of the different techniques so that we can address any misconceptions or problems which may arise.

Each teacher delivered the lesson to their groups.

Number of Teacher-researchers	Number of Classes	Total number of students participating in research
4	7	125

During the face-to-face lessons we took photos of students' work. Examples of students' work are included in the appendix.

Some of these lessons had to be delivered online which meant we had to use online tools to introduce the concepts. This meant that we could not assess students' understanding and responses fully, and that we could not collect as many examples of the diagrams students made by hand as we might otherwise have done in class.

After delivering the lesson we reflected informally in the shared workroom and formally in writing and discussion. As part of this process, we discussed how the lesson could be improved for next teaching.

As well as collecting examples of student work in the lesson, we conducted a survey a week later with the students to establish how much they had retained and what they thought of the approach. We used Google Forms for this so that the data could be held securely and confidentially. We also did a review of students' responses to the first formal assessment to see if students had used the diagrams and to what extent they had been successful. We analysed the survey responses to see what interesting features we could find. This helped us to plan for future lessons by adapting these schemes of learning for units next year, adding slides with extra activities.

Ethical considerations: All data collected from students has been anonymised in this report. We have reason to believe these approaches represent a good or better course than the students would otherwise have experienced, we are now attempting to evaluate the impact of using Bar models and Double Number Lines to teach these key topics.

Results and Discussion

A summary of the evidence from the surveys. Only maths questions got marks. DNL stands for Double Number Line. As shown in the table below, students averaged a mark of over 50% in each topic where DNLs or bar models would be useful, but the latter got notably higher scores: 72% for ratio and 66% for solving equations.

Topic	Students responding to survey	Average mark %
Proportion (DNLs)	28	52%
Ratio (Bar models)	58	72%
Speed, distance & time (DNLs)	53	51%
Solving Equations (Bar models)	89	66%

One concern about teaching students on GCSE resit maths using diagrams was that they might feel patronised or that the diagram would not be compatible with prior learning. For the majority of students surveyed this was not the case.

<u>Proportion using DNLs</u>

<u>Teacher's reflections on teaching the Proportion lessons using Double Number Lines</u>

Recipes lesson – this mainly used ratio tables not DNLs, but we did discuss multipliers/ dividing / scale factors. Students were able to engage with the context and relate it to their experiences. They had ideas and were able to contribute successful approaches. One teacher explicitly discussed HCF as a key idea for 2 step methods without a calculator, but this wasn't consistent across the implementation because the materials were very open.

Best Buys lesson – this was less relatable, students mostly wanted to buy the cheaper pack because they did not want so many pencils. After teaching this for the first time two slides were added to show the LCM vs cost per unit methods after having a productive discussion with one of the classes along this line. This tied in well with a Desmos activity on best value which emphasised the effectiveness of the cost per unit strategy. In the lesson students seemed keen on cost per unit as the "correct" approach. The cost per unit was an easy approach once it was taught using the double number lines.

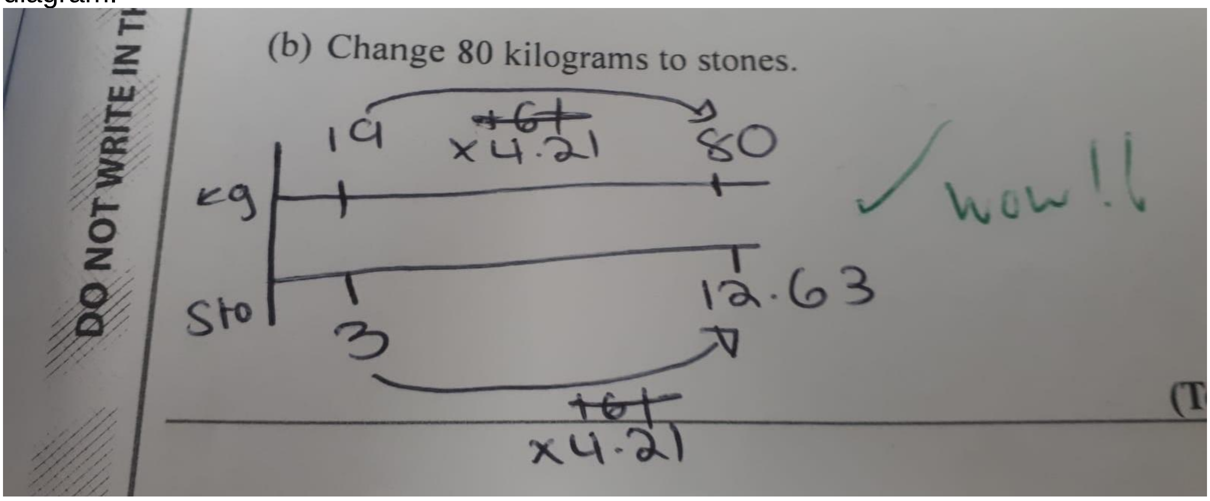
Scale diagrams & maps lesson – students did pretty well on the scale drawing – better than was expected from students at this level. The examples related well to the task which used elements of variation theory. Students favour multiplication when thinking about scale diagrams – they were more likely to talk about "How many 5s to make 20" than "20 divided by 5" The DNL for the tree was a bit confusing – we need to use separate DNL diagrams for each method rather than trying to show lots on one.

<u>Assessment examples</u>

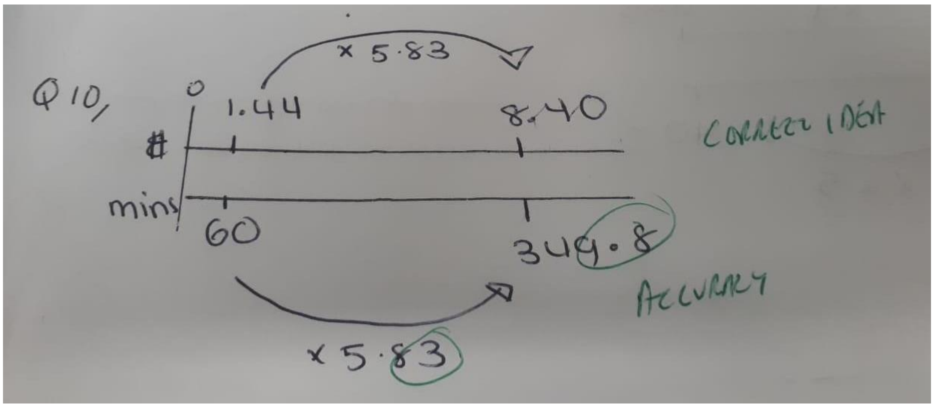
Students were encouraged to use the DNL to solve a range of exam style questions. Students also completed a key assessment from a GCSE past paper along with an extra sheet with blank DNLs and the question numbers where it could be used. There was also a hint to use a bar model in one of the questions. Students were encouraged before sitting the assessment to try the specific questions using DNLs. We found that 8/16 used a DNL or bar model at least once. 5 students used them more than once and most students who tried the bar model had full or partial success. Two students took the bigger part of the ratio to be the total number of boxes. Students who were partially successful had issues completing the DNL correctly in different situations. One student was still trying out multiplications to find the multiplying factor rather than dividing. Those who tried to use DNLs but struggled to get anywhere needed to work on labelling the lines and identifying a starting pair of numbers (£4500 is 100%).

Initially we intended to collect evidence of students' work from the topic assessments and key assessments we were doing as part of the course anyway. This had to be changed because the need for remote learning reduced the amount of student work we could see and changed the way we did assessments during the course. Here is some of the students' work we collected from the early assessments.

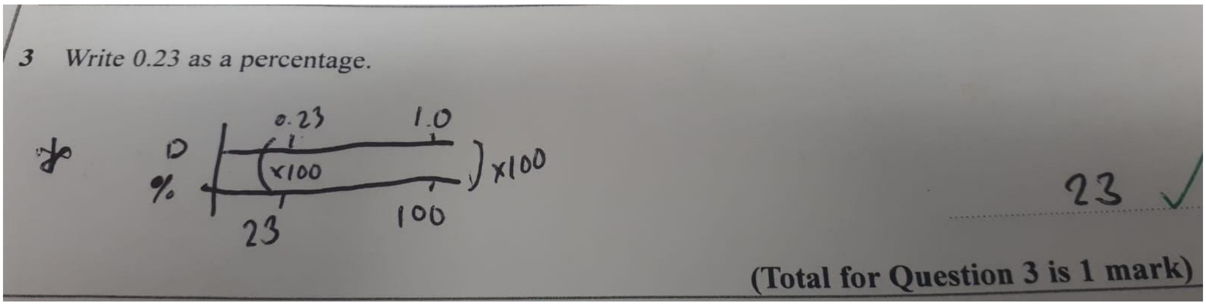
A good example: Here is an example of a student using a DNL on their own initiative for a unit conversion question in the key assessment (a past paper done in exam conditions.) This was the gold standard for us, very few people reached this level of confidence. You can see that the student has clearly labelled the lines in their diagram, they have calculated a scale factor from 19 to 80 and they are using the same scale factor to work along the lines in the diagram.



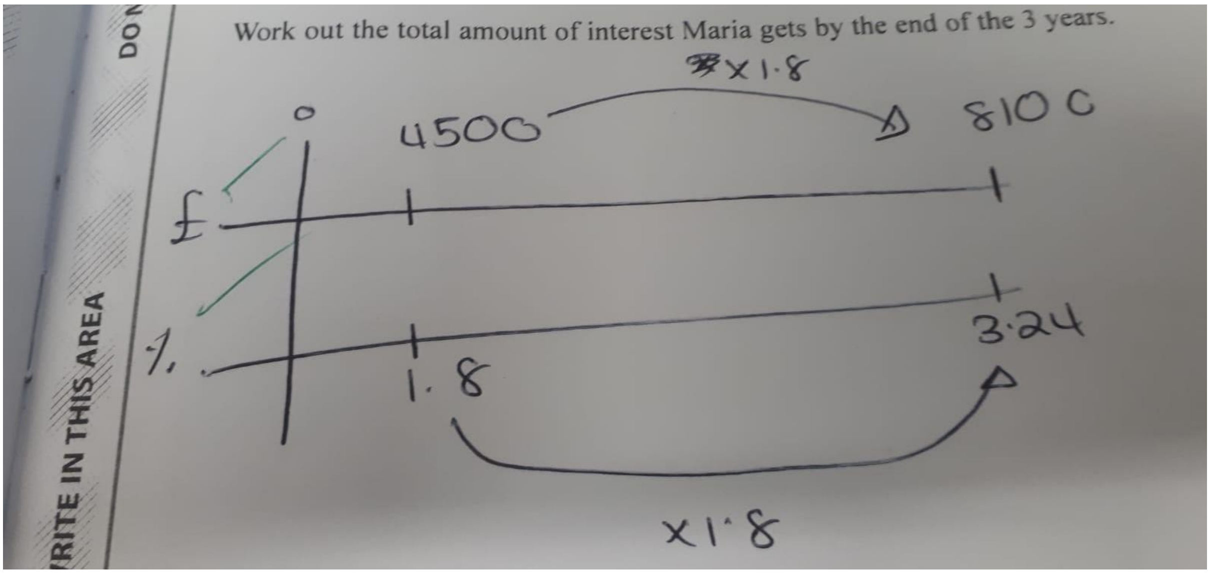
Accuracy: This student is trying a similar approach to work out a different proportion problem. Again, they have worked out a scale factor, but it has a recurring decimal and they have rounded rather than use the exact value. This has led to a loss of accuracy.



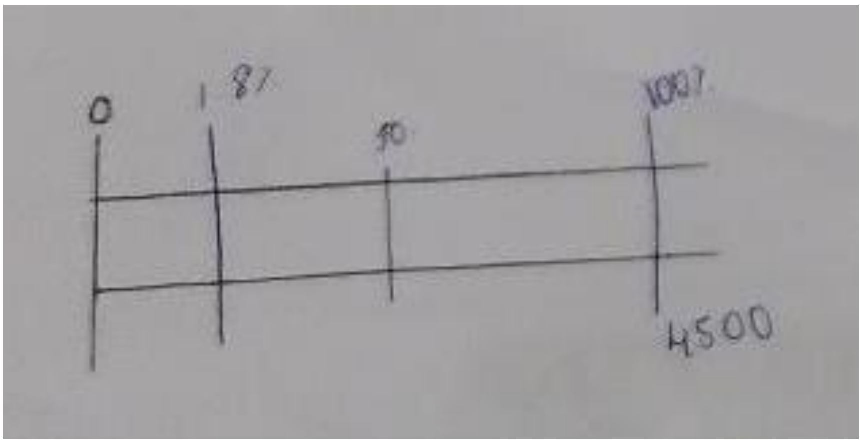
This student has successfully used a DNL to convert a decimal to a percentage. The two lines are not well labelled, but we can infer that the top one is decimal form and the lower one is for percentages. This student is working vertically between the lines as you can see from the curved lines indicating multiplication by 100 but it is not clear whether they are working up or down.



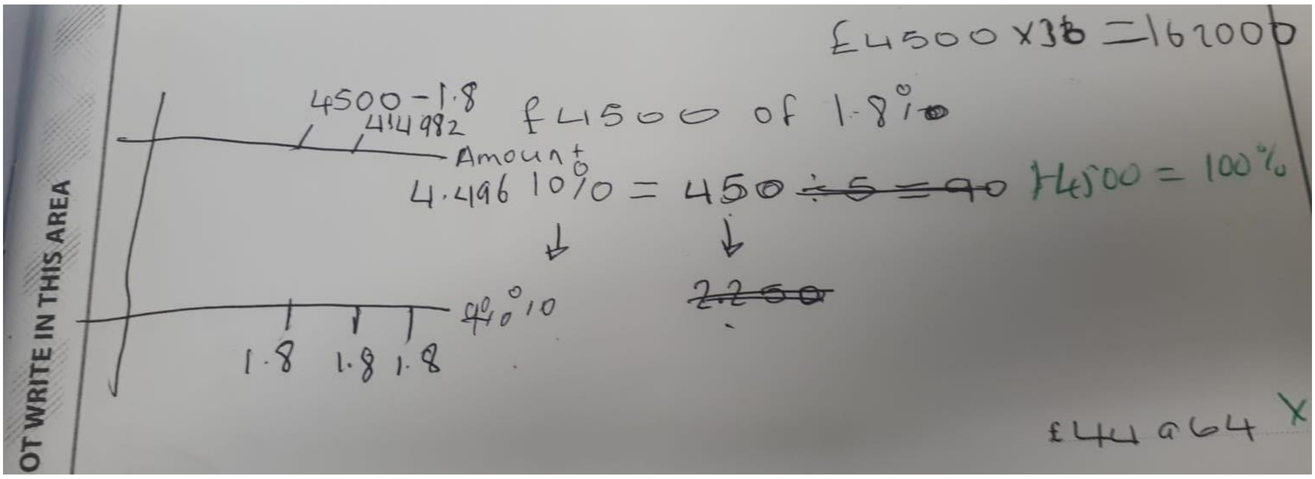
Using the wrong scale factor: This student is attempting to use a DNL to support working out 1.8% of 4500, but they have set up £4500 and 1.8% as the initial pair rather than 4500 and 100% Because they don't have a pair of values to find a scale factor they have chosen a decimal number from the question and used that instead.



Not being able to work out a scale factor: This diagram shows a better initial set up for the same question, however the student has not labelled the lines and has not been able to work out a scale factor going along or between the lines.



Retaining an additive strategy: This student has recognised that a DNL might be helpful but is using an additive strategy they may have learned earlier in their career which does not combine well with the DNL



Analysis of Proportion survey

28 students responded to the survey, although not all of them finished all the questions. All of them had some memory of using DNLs in learning about proportion.

The survey adapted 4 GCSE exam questions on proportion (a total of 10 marks). One has been omitted because it does not relate directly to the intervention lesson.

Limitations of the technology: Because the survey was made using Google forms to allow it to be done online as part of a remote lesson, we did not get any students' own drawings of DNL diagrams. Instead DNL diagrams with the key information in place were provided for students to use to work out the missing number and students were asked to explain the strategies they used with the diagrams. This does mean that any issues with setting up the DNL will not have been detected. However, it does allow us to classify the strategies more easily.

Recipe question: this was done well. Almost everyone was confident to attempt the recipe question. Most students (77%) said that they used the DNL diagrams to solve the problem and those who did had a significantly better success rate (81%) than those who did not (67%)

Q How much ginger/butter/sugar does Byron need for 40 gingerbread men?				
	Correct with diagram	Incorrect with diagram	Correct without diagram	Incorrect without diagram
No. of students	17	4	4	2

The majority of students (61%) said that they scaled up the recipe by multiplying by 2.5, which is represented by working along the line. This was a highly successful strategy with 93% success rate. The unit strategy (working out the amount for one then multiplying by 40) was less popular and also less successful. This is represented by working between the lines on a DNL.

Only 2 students chose a 2-step strategy (halve the amounts and then multiply by 5) and they were not successful. This could be a problem for the non-calculator exam where this kind of strategy might be important. Students did use this kind of strategy in the lesson with the ratio tables, but they had already worked out the smaller amounts – they didn't need to do it just for the larger amount.

	Cor/ x2.5	Inc / x 2.5	Cor /x 40	Inc / x40	Cor / x5	Inc / x5
No of students	13	1	4	3	0	2
strategy	scale factor (along)		unit method (between)		two-stage	
uptake	61%		30%		9%	
success	93%		57%		0%	

Scale Models: 24 students attempted the scale models question. Again, the majority (79%) said that they used the DNL diagram and those who did had a much better success rate than those who did not.

Work out the length of the scale model of the car				
	Correct with Diagram	Incorrect with diagram	Correct without diagram	Incorrect without diagram
No. of students	15	4	3	2
Success rate	with diagram 79%		without 60%	

The students who shared their strategy mostly preferred to work along the lines, recognising that 30 x 12 = 360 and scaling up by multiplying by 12, rather than using the scale factor given in the question and dividing by 30 (this would be represented as working between the lines.)

	Cor/ x12	Inc / x 12	Cor /div 30	Inc / div 30
No of students	11	1	4	3
Strategy	along lines		between lines	

uptake	63%	37%
success	92%	57%

Working along the lines was much more successful than working between the lines and I don't think the students would have spotted this scale factor if the information had been represented differently, for example using equivalent ratios like this 1 : 30 = ? : 360

Best Value: This question modelled the use of DNLs to work out best value using two different methods which had been discussed in the class for similar problems. The two strategies were LCM and unit cost. Students were asked to complete some calculations for BOTH methods and then asked to draw a conclusion about which pack was best value and say which method they would use or show their working.

	Unit	LCM	Other
Students who said they preferred this strategy	13	3	5
Students who attempted this strategy	20	23	3
% who got a correct value for comparison	70%	56%	

Unlike the other two questions, students said that they preferred working out a unit cost for the best value problem, although this is not borne out in the number of people who actually answered the related survey question.

21 students offered a conclusion, 4 of them without any correct working on either of the two methods. One of them appears to have just guessed (or used their life experience to intuit that the bigger pack is probably better value) but the others describe dividing price by amount – working out a unit cost, even though they got it wrong on the DNL.

	both DNL methods correct	1 DNL method correct	No DNL or wrong DNL	Total
correct conclusion	6	6	3	15
wrong conclusion	4	1	1	6
Total	10	7	4	21

Success rate	60%	86%	75%	71%

Surprisingly, the students who were successful at working out the calculations on both methods were not the most successful at interpreting the results and drawing the right conclusion. It seems like students who could only use one method for comparison have a better understanding of what their calculation might mean.

Ratio using Bar Models

Teacher reflections

At the start of the lesson a basic ratio dividing question was displayed on the board and students were asked to attempt the question. The students' work and responses were observed and noted.

Many of the students could not remember the different steps involved in answering the question.

The teacher then went through the question using both the traditional method and the Bar Modelling method. The teacher then went through another two examples using both methods. The students were then given questions to complete, the teacher used this time to provide one to one support and to give students feedback.

The weaker students seemed to adapt to the bar model quicker than the more able students, as they did not have a set method for working the questions out. The students found the process easier when the bar models were provided for them.

Students were able to apply the dividing method easily and then complete the rest of the questions using the bar models. Almost all students could answer the straightforward questions using the bar models, but the main problem was that the students were not able to identify the type of ratio question it was.

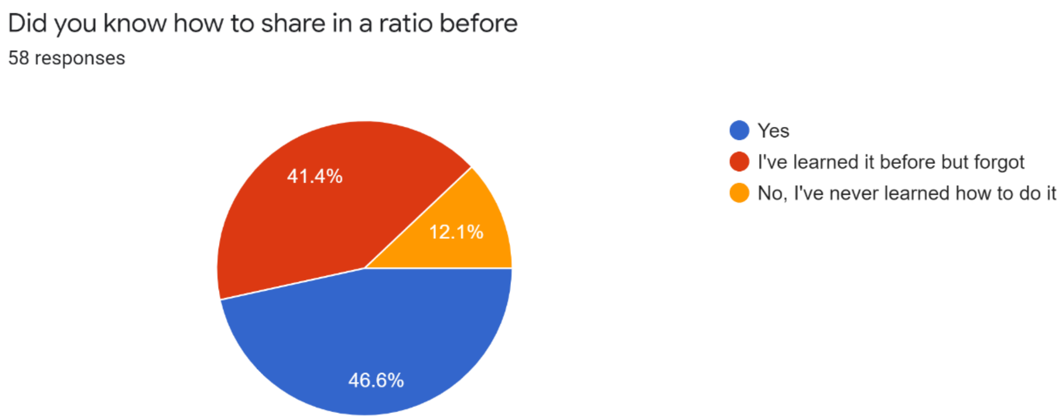
Some students were reluctant to try a different approach, so they were trying to stick to the method they were already comfortable with. Students need more practice with the bar modelling to ensure that they are totally comfortable with the concept.

Some of the students stated that they now understood the ratios a “bit better.” It was decided that the topic would be revisited in a few weeks and students assessed to see how much of the bar modelling method they retained.

Ratio using Bar Models - evidence from students work

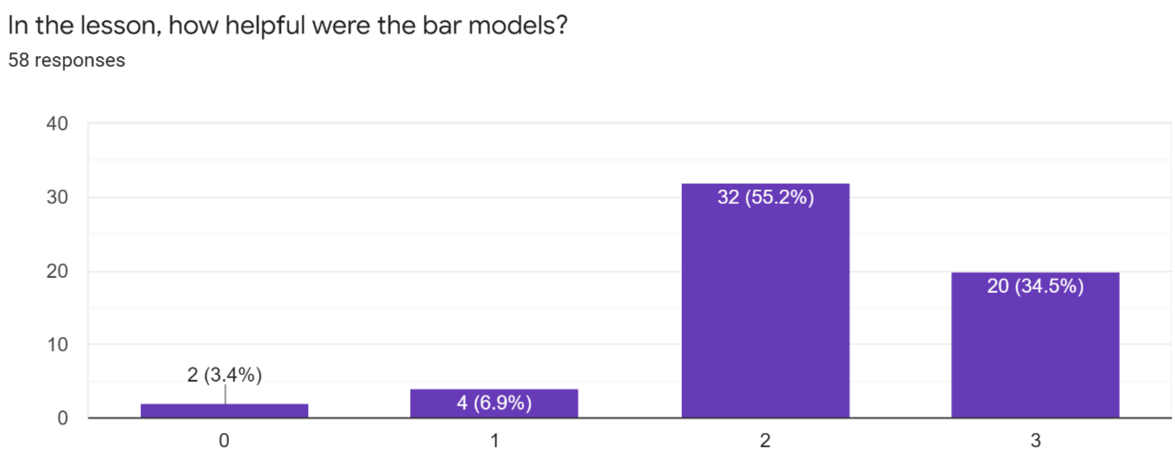
We collected evidence from students in 2 ways: a survey and a formal assessment (past paper Edexcel 2019 2F)

From the student survey it was clear that about half the students already had a clear understanding of dividing in a ratio.



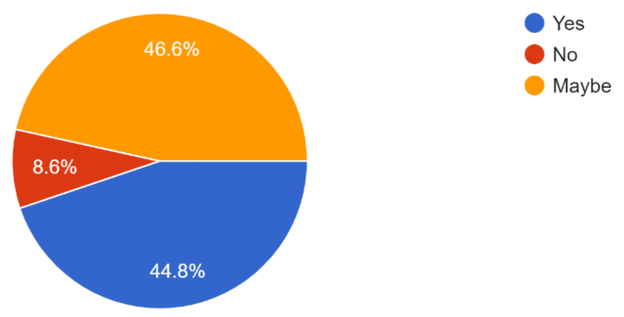
Once the bar models were taught the students who did not have an understanding of ratio previously found the bar models easier to adapt to.

In the survey students were generally positive about bar models, the majority (89.7%) considered that they had been useful in the lesson.

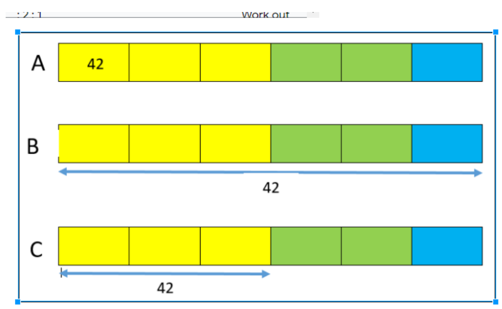


Nearly all students (91.4%) said they would at least consider using them for ratio questions in future as can be seen in the pie chart.

Would you use a bar model in the future to help you with ratio questions?
58 responses

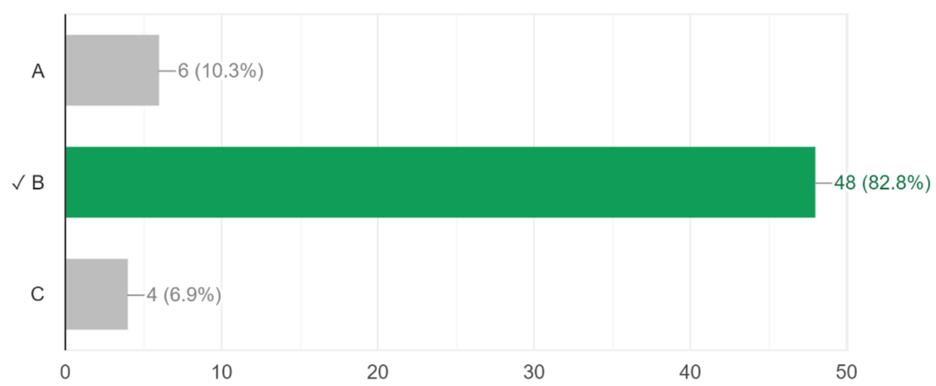


Students were given the following bar models



And asked

Which bar model is correct to help solve this problem? Molly, Paige, and Demi share 42 sweets in the ratio 3 : 2 : 1 Work out the number of sweets that each of them receives.
48 / 58 correct responses

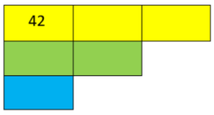


Most students were able to choose the correct bar model to solve the problem, and more than half (58.6%) successfully solved it. This compares well to 46.6% who said they knew how to share in a ratio from before.

The students were then given a stacked bar model

D

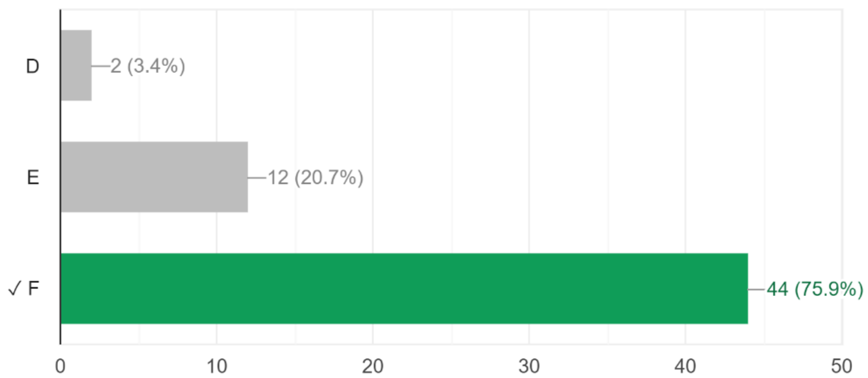
E



F

Which stacked bar model is correct to help solve this problem? Molly, Paige, and Demi share 42 sweets in the ratio 3 : 2 : 1 Work out the number of sweets that each of them receives.

44 / 58 correct responses



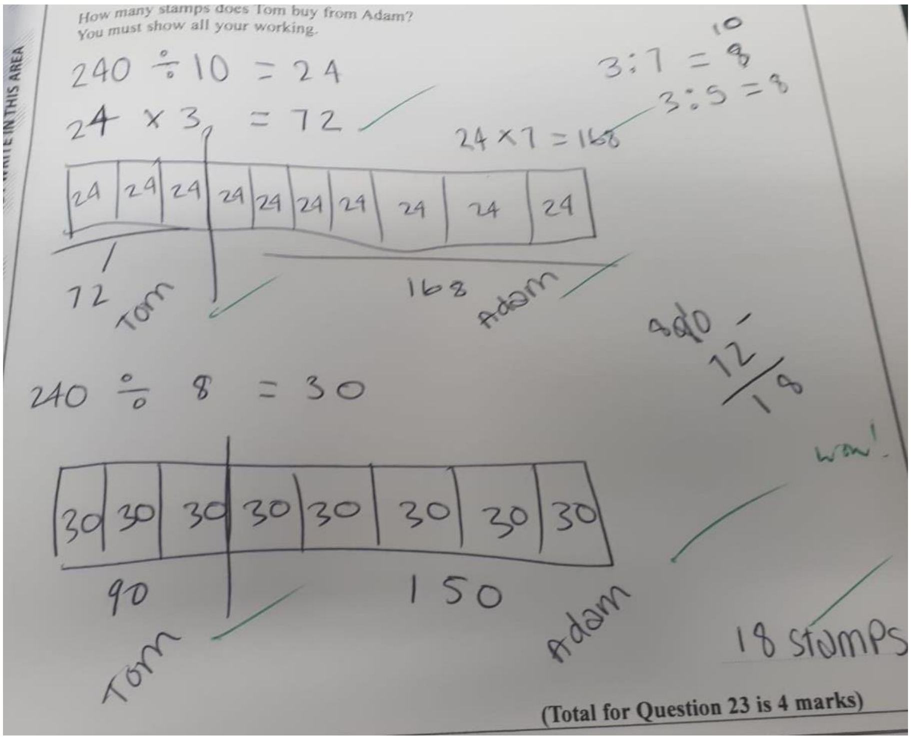
The majority of students were able to select the correct model. So we concluded that the students had enough understanding to choose the correct bar model if the models were given to them, but we then looked at the formal assessment to ascertain if students were able to apply the bar model correctly when the bar models had to be drawn by the students themselves.

However, on the formal assessment that we did around the same time, only a minority (22%) of students actually used a bar model for the ratio question (Q23). Of those who tried the question using bar models, only 1/12 got no marks at all, so this was a successful strategy. The Examiners report for this question suggests that across the cohort many students were able to make a start and get some marks on this, but we are focused on students who got grade 2 or below here.

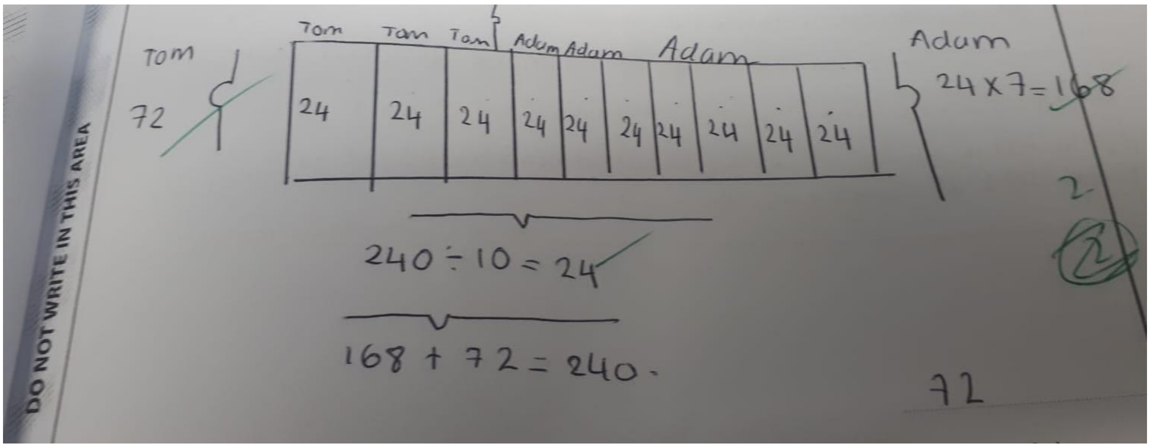
Table: a review of q23 on 50 scripts from the formal November assessment

	some or all marks	no marks	total	success rate
Bar model	11	1	12	92%
No bar model	10	9	19	53%
No attempt		19	19	0%
Total	21	29	50	42%

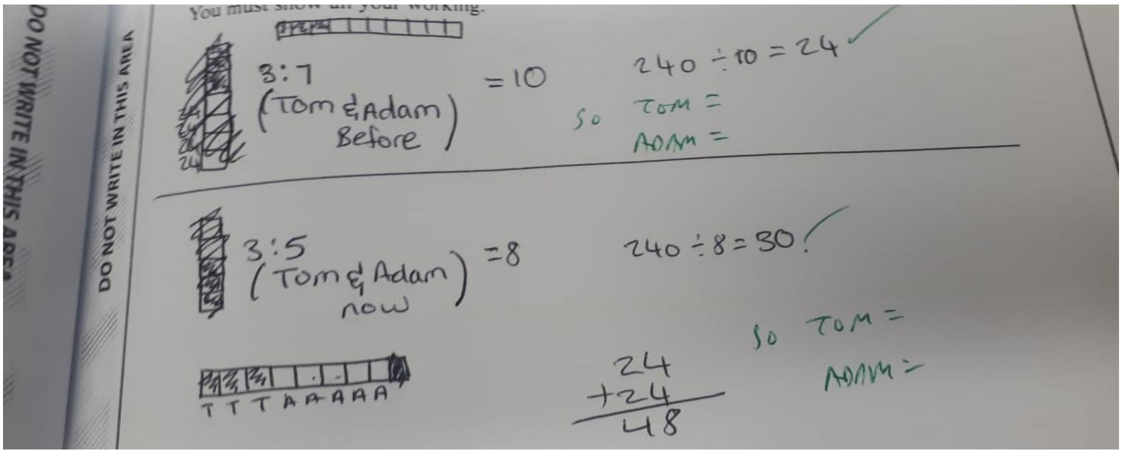
The following student applied the bar model correctly and received full marks for the question.

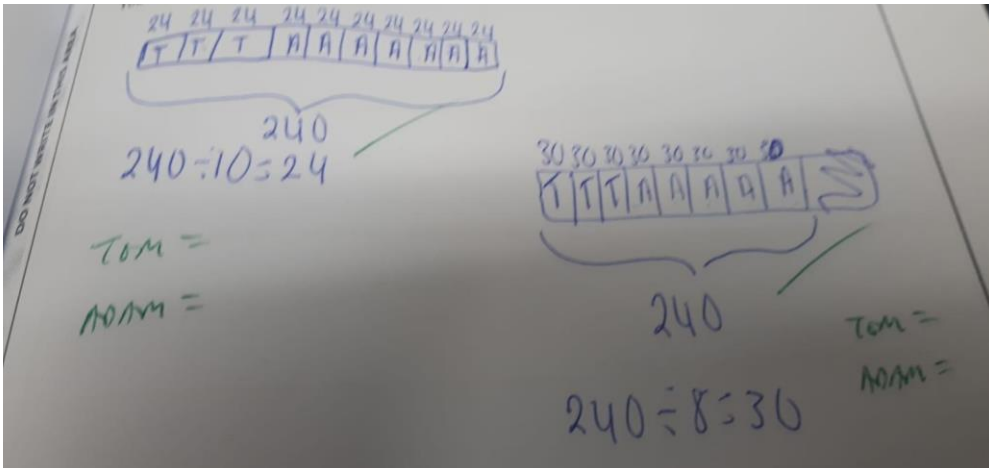


Below are some examples of students' work where they received some of the marks for the question.



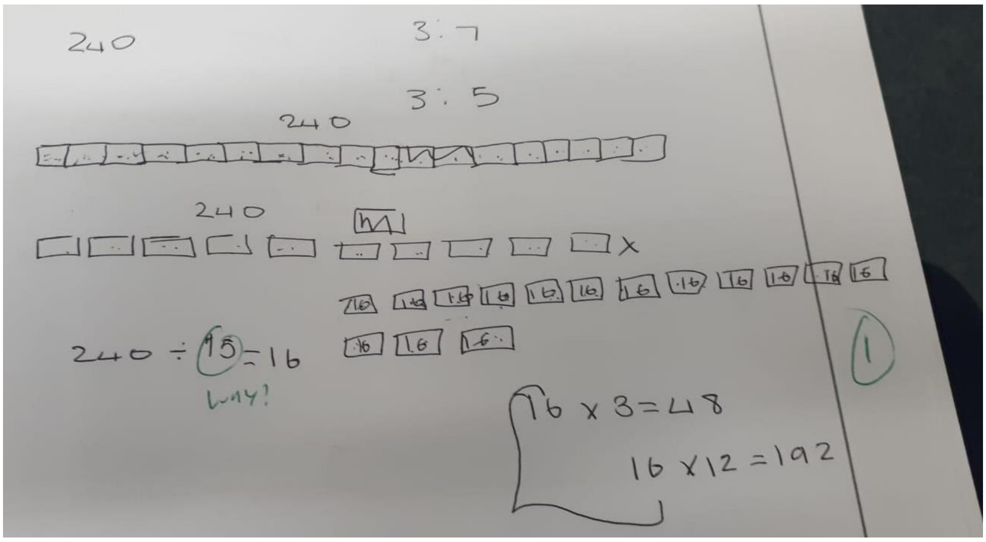
This student has drawn the bar model and has then split the ratio correctly, he has then calculated how much each person receives but then has not gone onto complete the second part of the question.

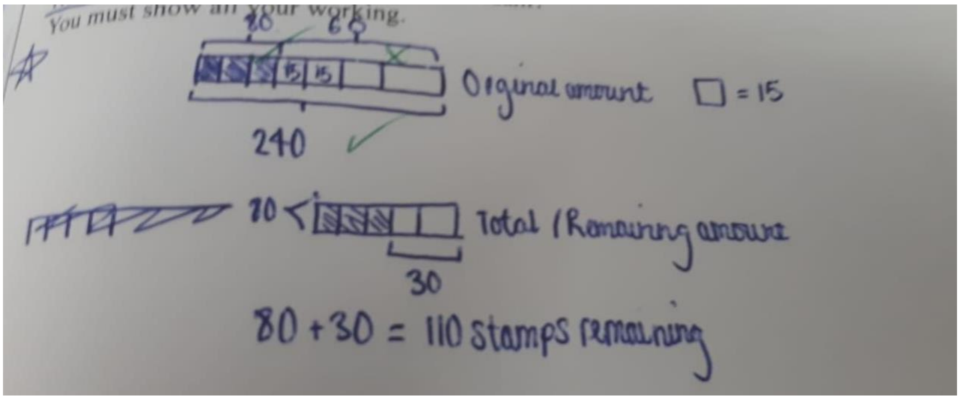




Both these students have drawn the bar models for both parts of the question and then have gone on to calculate the value of each part but have not completed the question.

The following students attempted the questions but failed to set the bar models correctly.





Why did students fail to transfer a popular and highly successful strategy? We think perhaps it is because in the survey and the lesson the bar models were drawn for them. This scaffold was not there in the formal assessment. We need to support students in the transition to using this strategy independently during lesson time. Once they can use a pre-drawn model, students need to be pushed to draw their own bar models and to use this strategy on exam questions which are formatted similarly to those in the assessment in order to recognise that they should transfer this in the context of an exam.

Speed Distance & Time using DNLs

Teacher reflections

We taught Speed Distance and Time using DNLs to understand the connections between the three. All DNL diagrams had 2 lines, one for distance (top) and one for time (bottom) to reflect the shape of the speed = distance / time formula and to give students confidence in creating their own diagrams. We started with a video about the fastest animals which gave a **concrete** element which students could connect to things they've experienced or seen and then went on to represent the situation **pictorially** with DNLs. Some students had also learned the speed = distance / time formula so we also were able to connect the calculations on the DNLs to this more **abstract** layer.

Students did seem to definitely get the idea of speed in mph as "the distance you would travel in 1 hour" which is good because often they seem a bit haphazard on units in these questions. This carried on well into later work with Distance Time graphs. The DNLs connected well to the formulas – we used 3 versions rather than a formula triangle and I saw fewer mistakes in dividing the wrong way round or dividing instead of multiplying etc. Students found the DNLs especially helpful in questions which involved more proportional reasoning than simple substitution, especially with minutes.

After the first teaching, I changed the materials a bit to make them more accessible to students accessing the lesson using smartphones. We created a topic check on past exam questions on speed and distance/time graphs and collected data via a Google Form.

Analysis of evidence from teaching speed distance time using Double Number Lines

This topic was taught remotely so we were not able to get photos or scans of students' own diagrams directly. Instead, we collected evidence using teacher reflections after teaching the trial lesson and through an online survey where students selected the correct diagram from 4 pre-drawn options.

From the teachers' reflections there were some common themes:

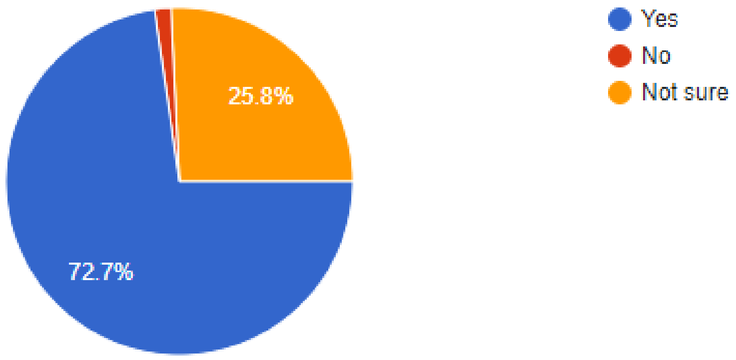
- Students were more confident in this lesson than we would generally expect for students at this level, and this may have been because the visual representation was easier to understand than a formula
- Students were confident in talking about speed as the distance travelled in 1 hour. This shows conceptual understanding
- Students were generally better at calculating speeds than distances or times
- Problems that involved converting between hours and minutes were very challenging.
- Students found it hard to put the information from the question into a DNL, but if it was done for them, they could use it to solve the problem

From this we can suggest some improvements to the lesson

- A starter activity on converting hours and minutes / time differences would help to secure prior knowledge before it was needed and reduce the cognitive load when the skill was required in a speed distance time problem
- More practice on distance and time questions is needed, perhaps keeping time in hours for longer
- It is important to spend time on making sure students' diagrams are correct and that they can produce them independently (although this would be easier in person.)

Did your teacher show you double number line diagrams as part of the examples in your speed, distance and time lessons?

66 responses



From the survey, we noticed that around 25% of students were not sure if they had used DNLs in the lesson, so it seems like that was not well retained. Students were better at calculating speed than distance, as we expected from our reflections, and they did reasonably well on the third problem which was about interpreting a distance time graph.

	Correct diagram and answer	Correct diagram, wrong answer	Wrong diagram, correct answer	Wrong diagram, wrong answer
Calculate speed from distance & time (45% chose the right diagram)	28	2	29	7
Success rate	with diagram 93%		without diagram 81%	

Calculate distance from speed & time (21% chose the right diagram)	9	5	18	34
Success rate	with diagram 64%		without diagram 35%	

There were 2 problems for which students had to select a diagram and then solve the problem and it is interesting to see that in both of them at least as many students got the right answer without being able to choose the right diagram. Clearly, they have other methods of solving the problem, and perhaps DNLs are not a good fit for their own mental picture of what's going on.

The results on the distance time graph section were good (average 59%) which suggests that students are much more able to make sense of distance time graphs than they are of DNL diagrams for speed, distance and time. Next time I teach this topic I intend to use sketched distance time graphs instead of DNLs – essentially a perpendicular DNL – but this approach has not been tested as part of this research project.

Solving equations using bar models

Teacher reflections

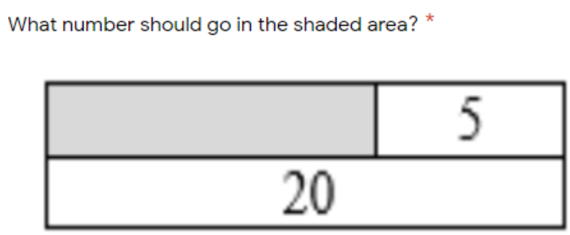
- Starting by understanding the bar model diagrams with basic arithmetic was useful
- The mobile phone plan task was engaging because it was initially very open and so students could express their opinion without being wrong, so that when it progressed to a more challenging closed question, they were feeling confident
- For some students, being able to visualise an equation was a breakthrough
- The teacher needs to be absolutely committed to the bar model to get students to engage with it. If it is optional, it will not happen and nor will those breakthroughs
- The bar models work best with one step equations especially multiplying or adding – division and subtraction were more challenging
- The mini-whiteboard formative assessment session was essential for students to build skill and confidence drawing their own bar models for questions that did not come with one
- Showing balance method alongside the bar modelling was important to support students making connections with the method they have seen before and towards progression to a fully abstract method

Analysis of the Equations survey results

89 students completed the survey. The majority remembered using bar models in the lesson.

Most students (75%) considered that the bar models had been helpful in the lesson.

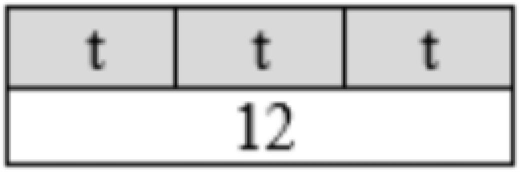
There were 4 questions assessing the students' use of bar models and skill in solving equations.



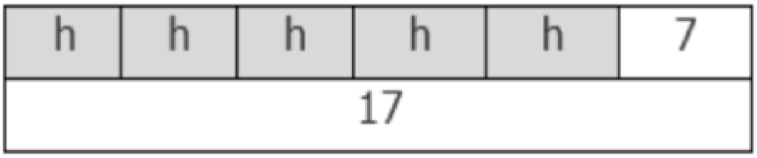
This question was intended to assess the students' understanding of the bar model although it is equivalent to the equation x+5=20

The most popular response was 15, which is correct (61% of students), and the most common misconception was 4 (4x5=20) which may point to a confusion between x and + or may indicate a poor understanding of the diagram.

Use the bar model to solve 3t = 12 *



This was well done with 75% of students getting the correct solution t = 4. The most common misconception was t=3 which means the students were not confused between x and + at this point but they may be confused between the value of t and the quantity of t.



Students were asked to identify which equation this bar model represented and then to solve for h.

76% could select the correct equation (5h+7 = 17) which suggests they have a good understanding of this representation of an equation but only 53% correctly solved this 2-step equation to get h=2

The most common misconception was h=10 which comes from just doing 1 step (subtracting 7) and is connected to the idea that the part of the bar which is shown by the hs is 10. 30% of students still had this misconception.

Only a few students (12%) said they would not at least consider drawing a bar model to help them solve an equation in the next assessment. Looking at the raw data and the names of specific students in question we noted that these were students that we had observed in the lesson were proficient in using the balance method and in some cases had found the bar model harder to understand. Their success rate in solving the 2-step equation was 64% so they do have a functional strategy, so it is not unreasonable for them to stick with it rather than adopt a new one. It’s important to reiterate from the teacher's reflections that it is still important to make everyone try the new method even if some students will eventually choose a different one.

Conclusions and Recommendations

1) **DNLs and bar models can be used to support reasoning in a wide range of GCSE maths exam questions including ratio, proportion, fractions, percentages, compound measures and equations.**

Teachers need to be prepared using CPD or group practice sessions to know how to use the diagrams themselves in order to use these effectively in class so that they can be confident in delivering lessons using this approach.

Moving topics that will be taught using diagrams earlier in the scheme of work establishes the expectation that drawing a diagram is part of the mathematical process and increases the time students have to practice using the diagrams in a range of contexts through the year.

2) **It is worth taking the time to teach students how to use bar models and DNL diagrams, making sure that they are drawing them properly and insisting that they draw the diagram alongside their working when they do the practice questions.**

Explicitly teaching students to use the model with very simple mathematics prepares them to use the diagram to build their conceptual understanding of the main topic of the lesson. Reinforcing the process of constructing the diagram makes them more likely to attempt to use it when solving exam questions that do not come with a prepared diagram.

Activities where students match up and organise different representations of the same worded or abstract problem are valuable so that when they see a question in an exam, they are able to make the connection to the pictorial representation and then solve it.

B A phone company charges a 10p connection charge and 4p per minute for UK calls. If Julie is charged 90p how long was the call?	m | m | m | m | 10 90	
C Sarah pays £10 a month for her mobile phone contract and is charged extra for data at £6 per gigabyte. If her bill for the month is £28 how many gigabytes of data has she used?		6d + 10 = 28

Doing exam style questions in class with the DNL or Bar Model also makes it clear that students can and should use these in the assessment to aid them in tackling questions they would otherwise avoid.

3) **Students who do not have a method or who struggle with traditional methods are more likely to attempt to solve a problem if they have learned how to picture the information on a diagram.**

Having experimented using these approaches with students with very low prior attainment, we intend to expand the use of these teaching approaches with students on

GCSE resit with higher levels of prior attainment. We will be showing these representations alongside the more traditional methods that students may have learned before so that they can make the link. We believe these diagrams will be valuable to support them in explaining their reasoning and making connections to prior learning, variations on the “standard” problem and other areas of maths.

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Appendix/Appendices

Lesson Resources

Copy of Proportion 2 Lesson.pptx

Copy of Ratio using bar models.pptx

Copy of Speed Distance Time (original version).pptx

Copy of Equations using bar models v2.pptx

Copy of Equations using bar models - worksheet V2.docx

Student Surveys

Copy of Proportion 2 - student survey

Copy of Speed, Distance & Time - student survey

Copy of Equations with bar models - student survey