Effective use of representations and manipulatives to support learners understanding with groups containing some or all Grade 3 students

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OUR PARTNERS

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About CfEM

Centres for Excellence in Maths (CfEM) is a five-year national improvement programme aimed at delivering sustained improvements in maths outcomes for 16–19-year-olds, up to Level 2, in post-16 settings.

Funded by the Department for Education and delivered by the Education and Training Foundation, the programme is exploring what works for teachers and students, embedding related CPD and good practice, and building networks of maths professionals in colleges.

Summary

The issue we were trying to address was how to improve engagement for grade 3 students with topics that they previous found difficult. We also wanted to develop students conceptual understanding within these topics so that they would be better equipped to answer a range of questions.

To achieve these aims we used representations and manipulatives to support understanding within solving equations, compound measures and percentages.

We collaboratively designed and taught lessons on these topics using different concrete or pictorial approaches and collected data from student quizzes before and after the teaching. As we also wanted to see how using these approaches worked for staff new to them and also what issues staff had implementing this with students with a range of abilities, we collected teacher reflections and some student feedback.

Our main findings were that all the representations and manipulatives we tried are useful to support some students develop their understanding but we found that Double Number Lines were particularly successful at this but also as a structure for students to solve problems. They were the most flexible representation as they could be used across a range of topics and they supported progression through to more difficult questions.

In terms of how to introduce and use these approaches we found that it was best if they were used early on in the course and consistently. We also found a flexible teaching model was essential to engage students with these approaches. Students wanted to use their own methods where they were confident to but having been shown the representation alongside this, they often starting using these for trickier questions. We found that starting with trickier questions helped get students on board with this new approach.

Contents

Background

This action research project was conducted by five teachers across four colleges.

Our colleges and cohorts

Barnet and Southgate College is a large college with over 13,000 students spread across 4 campuses. We offer a large range of Vocational courses, A-Levels and a GCSE course. We also have large ESOL and LLDD departments.

BSix is a comprehensive sixth form college in Hackney, East London. Entry requirements for Level 3 qualifications are amongst the lowest in the sixth form sector. Approximately 93 per cent of students come from Black, Asian and Minority Ethnic (BME) groups, and around 23 per cent are eligible for Free School Meals (FSM). The College has approx. 1100 students of which, 26 per cent come from European countries and 9 per cent have studied overseas prior to joining the College.

Leyton Sixth Form College has about 2000 students, mostly aged 16-19 and studying full time at level 3. Around 60% of students are doing A-levels and 40% are on vocational programmes such as BTEC. We also offer BTEC and ESOL courses at Level 1 and 2 to enable students to access further learning through progression at the college.

Stanmore College has about 1000 students with the majority aged 16-18. The Maths department is within the school of STEM.

College	Borough	Index of Deprivation (Out of 326 Boroughs)
BSix	Hackney	78th most deprived
Barnet and Southgate	Enfield	74th
Leyton	Waltham Forest	82nd
Stanmore	Harrow	207th

Our learners and our GCSE Maths courses

Barnet and Southgate have historically only entered Grade 3s for GCSE, however we have changed that policy this year and are not running the level 2 FS but automatically placing students onto the GCSE course. This decision was taken as the pass rates for Level 2 FS were disappointing and it was felt that achieving a 3 at GCSE level would be a better step towards achieving a 4. This means we have 1000 GCSE students this year compared to about 800 usually.

At BSix the number of GCSE maths starters range from 360 to 500 students. We offer GCSE Maths to all students who have not yet achieved a grade 4. We split this cohort into Foundation tier and a small Higher tier group, and for the last two years our Foundation groups have had students ranging from grade 1 – 3. For several years our Foundation cohort has been broadly grouped depending on their Core study programme.

Leyton has about 550 GCSE Maths students each year. We offer GCSE Maths to all students who have not yet achieved a grade 4, splitting our groups into Grade 3 and Grade

0-2 learners. The Grade 0-2 SoW has been recently developed to better progress students to grade 3.

Stanmore has about 550 students including adults who are enrolled in Maths GCSE. This includes a small proportion of learners who are at an off-site provision via subcontractors. GCSE maths forms about 52% of the overall students' enrolment.

The GCSE cohort is split into higher and lower groups which means that students with grade 3 and grade 2 on entry will be taught is separately while students with grade 1 and below will be entered in the Functional Skills Level 1 course.

Summary of our 16-18 GCSE results for the last 3 years for Grade 3 learners only

Year	Barnet and Southgate	Leyton	BSiX	Stanmore	All FE
18-19	19%	30%	23.4%		18%
19-20	29%	43%	26.9%		28%
20-21	38%	49%	39.5%		34%

Our project:

An action research project on the use of representations to support understanding was carried out by Leyton teachers in 2020-21. This was aimed at grade 0-2 students as part of a wider trial to apply mastery approaches within an FE setting.
The successful outcomes from that project have led us to introducing these approaches to our Grade 3 learners at Leyton. Partner colleges who had limited experience of these approaches, were also interested in trying out the use of representations in their own settings.
As such the aim of the project has been to see how using representations works with grade 3 learners in a range of college settings and class set-ups. Some classes were only grade 3 students whilst others were mixed ability.

Literature Review

Introduction

Student's experience of learning mathematics in this country involves an accelerating one-way march through the concrete and pictorial to the abstract. Particularly from the early years of Key Stage 3, students are assumed to have the understanding to develop familiar and new topics through the medium of signs, symbols, and algorithms. Although this is true for some students it becomes a real issue for learners who cannot make the connections to their previous understanding and struggle to hold the increasing amount of knowledge required in their working memory. Mark McCourt says in his book "Teaching for Mastery" (2019)

"It is important to recognise that the elements of the connective model do not form a hierarchy and it is not intended as a pathway from the concrete to the abstract. Rather we can move between the representations as appropriate. We want all pupils to be able to work with all mathematical ideas efficiently in the symbolic form but there is no rush here. Giving pupils many opportunities to examine a mathematical idea from a range of viewpoints enhances their appreciation of the underlying principles behind the idea."

Otherwise, this can lead to misconceptions that become entrenched and a growing lack of confidence amongst some learners. Post-16 GCSE resit students often have incorrect or inflexible models for solving mathematical questions or even no model leading to random applications of partly remembered rules.

PISA results from places such as Singapore has led to a growing movement within many primary and some secondary schools to examine and try out the teaching approaches used in these countries and there is already evidence of a small positive impact (Jerrim, Vignoles 2016). This approach recognises the importance of the Concrete-Pictorial-Abstract continuum and the need for students to move backwards and forwards within this as they develop their understanding. The role of representations as a key part of this approach has been developing for some time and our action research group was interested to see what the implications of using them would be within a one-year GCSE course, where time is tight, students come with very different experiences and mathematical approaches and staff have limited experience of using them. Our research group involves staff from four colleges most of whom have not had much experience of using a CPA approach in their classrooms before so part of the process is to develop our understanding of how manipulatives (objects that students can hold and manipulate) and mathematical representations (such as bar models and double number lines) work together and in isolation to support the learning process.

Manipulatives

A wealth of existing research highlights the importance of manipulatives in the concrete stage of the concrete-pictorial-abstract approach. Manipulatives can be any physical item that supports students understanding through the process of organising and reorganising these items to match the mathematical situation. Stein and Bovalino (2001) note that, "Manipulatives can be important tools in helping students to think and reason in more meaningful ways. By giving students concrete ways to compare and operate on quantities, such manipulatives as pattern blocks, tiles, and cubes can contribute to the development of well-grounded, interconnected understandings of mathematical ideas".

It is also understood that manipulatives support learners to link ideas and different areas of mathematics. Clements (1999) calls this "integrated concrete knowledge" and notes that manipulatives help students to connect and integrate knowledge.

In order to support learners to achieve this greater level of understanding, it is important to select the correct manipulative. Skevington (2016) underlines this, explaining that “Manipulatives that can be easily changed by learners to represent different concepts and mathematical ideas are the most valuable as they allow children to ‘bridge’ and transfer their understanding of different concepts.”

Research with a small group of Year 5 students by Williams (2018) showed that manipulatives clearly helped pupils to move past a surface understanding of concepts. Pupils who have struggled to engage fully in mathematics lessons found success for the first time and more able pupils enjoyed representing challenging concepts. He does add that,

“I had to spend time developing my professional knowledge. This has implications for other practitioners if they are going to move away from a more traditional style of teaching. I invested in many guides and books which provided ideas and strategies for introducing manipulatives”.

Further examples of the uses and possible pitfalls of manipulatives leading to representations are highlighted by the NCTEM KS3 Guidance for using Algebra Tiles. A main misconception in manipulating algebra, is incorrect use of directed numbers and as such a central idea in the use of algebra tiles is that of zero pairs (pairs of numbers that sum to zero). As such student’s familiarity with this idea with positive and negative numbers as a pre-requisite, is essential and students should be able “to spend adequate time establishing the concept of zero pairs with directed numbers first.” Once this is mastered students can confidently use the tiles to collect like terms and simplify expressions.

When using the tiles to solve equations, the concept of the x tile representing a variable is key and that the actual dimension of the tile is not related to any possible value the x might be. An online tool (Hooda Math Algebra Tiles) allows the x tile to be altered by a slider and it is suggested that this might help students “grasp the idea of x as a variable and support a move to the use of bar model diagrams, where the limitations of concrete tiles can be overcome by a more flexible, pictorial representation”. Again, the use of zero-sum pairs is crucial “when two sides of an equation do not contain any of the same tiles and the introduction of a zero-sum pair can be applied to the solution process”.

One advantage of algebra tiles over other representations (including bar models) is that they can present a way of representing a negative solution. However earlier work would be required to help students understand that flipping the tiles is equivalent to multiplying by -1, so that visually $-x = 3$, can be changed to $x = -3$.

They suggest that algebra tiles always be used alongside symbolic notations so that when the tiles become cumbersome, students can “switch” to algebra, implicitly suggesting no need for a further pictorial stage in between. This has the implication that using algebra tiles or bar models to support understanding with equations can be useful but maybe not both.

Bar Model Representations

Representations are examples of the pictorial stage in the concrete-pictorial-abstract approach. There is plenty of research at primary and secondary level on the use of bar models to support understanding both in the sense of giving a visual representation of the problem and highlighting potential ways of solving the problem. Spencer’s et al’s (2015) work with KS2 students on word problems involving fractions and all four operations including two-step problems, showed that children who are normally confident with calculation often experience difficulties with the interpretation of word problems. The use of a bar model was valued by the children more in areas of mathematics that were new to them and where they felt less confident.

This is a key point for teaching the GCSE retake course because as Spencer say:

“There was a clash between teachers trying to teach it, starting with the basic addition and subtraction models, and the students finding this frustrating because they could work out the answers without needing to use it. However, without the teachers introducing it from ‘the basics’, the children wouldn’t have necessarily had the appropriate bar modelling skills to attempt more difficult areas of mathematics.

A study of secondary school students in India by Thirunavukkaras (2017) used bar models to teach a unit of algebra. Pre and post tests showed a significant improvement for those students in the bar model group compared to the control group. There was no significant difference of progress within the bar model group between students with outside tutors and those who didn't or with those rating maths as their best subject and those who didn't. This suggests a potential benefit to using a representation to support understanding in an isolated topic.

Again, the NCTEM’s KS3 Guidance, this time on using bar models, summarises lots of experience of using this representation in the classroom.

“Bar models can support students in deciding which calculations they need to perform and help them understand what to do in order to get to a solution. This understanding of the why and the how is essential for students when it comes to fluently applying their skills and knowledge to a wide range of mathematical problems. Students need to have a good understanding of key concepts when using bar models and once they can confidently use bar models as part of their reasoning process, they can apply the technique to many different areas of the maths curriculum.”

They also say that

“Bar models provide a bridge for students between concrete objects and abstract symbolic representations”

Looking specifically at solving equations using bar models, they state.

“While algebra tiles provide students with physical manipulatives that can be picked up and moved around, the concept of equality is shown less well using this representation. Bar model diagrams, in contrast, can depict equality between two expressions very well and in a way that reveals the equations' structure more readily”

However, they warn that

“This, however, is more problematic when $x<0$ and so the decision of when to use bar models to support students in understanding solutions to linear equations should be carefully considered. It is helpful to examine the symbolic algebra alongside the bar model diagram in order to relate the manipulation of the bars with the manipulation of the symbols”

So, it seems that the representation is there to reveal the structure and not to “get the answer”, the use of bar models alongside the algebraic symbols could support students in developing a deep understanding of additive and multiplicative relationships and their combined use when solving linear equations.

Finally, the Action Research project by Leyton Sixth Form teachers (2021) stated that

“DNLs and bar models can be used to support reasoning in a wide range of GCSE maths exam questions including ratio, proportion, fractions, percentages, compound measures and equations.”

And that.

“It is worth taking the time to teach students how to use bar models and DNL diagrams, making sure that they are drawing them properly and insisting that they draw the diagram alongside their working when they do the practice questions.”

But they highlighted that

“Students who do not have a method or who struggle with traditional methods are more likely to attempt to solve a problem if they have learned how to picture the information on a diagram.”

So, targeting our interventions at these students will be important, although it is not always clear without further teacher investigation, which students have a solid grasp of a method and how far that method takes them through a range of different types of questions.

Conclusion

So, to investigate the efficiency of representations and manipulatives for a greater ability range of students the literature review and last year’s project lead us to investigate the use of key representations such as bar models and the how this might be supplemented by or replaced by manipulatives in some topics such as algebra.

There is also the issue of how to convince students to take on a new learning tool, that would need to be demonstrated through simple examples before students could apply them to more realistically difficult situations.

Finally, we also need to balance our work with three groups of students in the same classroom. Those with no previous traditional methods, those who can use these traditional methods but in limited scenarios and those who can use them confidently. Further, this might be different for different topics.

Methods

We see action research as the collaborative process of reviewing our current practice in a particular area, establishing what the current research says about this and then undertaking several cycles of planning, intervention and review to improve that practice

This project is a follow on from one last year that was solely based at Leyton Sixth College and focussed on Grade 0-2 learners. This year's project had several aims:

- To see if the successes of last year could be replicated across different colleges
- To examine the similarities and differences when working solely with Grade 3 leaners or mixed classes containing Grade 3 learner.

Our outcomes are based on both qualitative (teacher reflections and student feedback) and quantitative data (test data)

We undertook 3 cycles where we identified a topic area that was common to each college's Scheme of Work that half-term, which could be supported with use of representations and manipulatives. We then collaboratively planned the activities and reflected on the outcomes.

Teachers from three of the colleges had limited or no experience of using the representations or manipulatives so we did some practice beforehand to increase our own confidence but using the activities in class was often a learning experience for us.

Teachers	Classes	Students
5	9	230

Autumn Half-Term 2:	Solving equations	Bar Models / Algebra Tiles
Spring Half-Term 1:	Speed	Double Number Lines
Spring Half-Term 2:	Percentages Revision	Double Number Lines

Each intervention consisted of a pre-teach test, a taught lesson (2 in the case of solving equations) and a post-teach test. Further feedback was collected through teacher reflections forms which also included some student feedback.

The tests were short online tests with diagnostic style multiple-choice answers. The post-teach test for percentages was done on paper. This was because we felt that some students might have been rushing through the previous online tests and also as we wanted to see how the students were using the DNLs at this stage.

The lessons were designed along a similar format of a PowerPoint to model the use of the representative to answer a range of questions on the topic, a matching exercise to develop students ability to understand and use the representative and then an application of knowledge activity that students the chance to see if they could use the representative to support their understanding with less scaffolding.

Each college has a slightly different lesson length and adapted the content of the activity to suit that and the range of abilities in their classes.

College	Lesson Length
Barnet and Southgate	1 Hour 30 mins
BSix	1 Hour 10 mins
Leyton	2 hours
Stanmore	1 Hour 30 mins

Students understood that they were taking part in an action research project that was aimed at improving their outcomes and data collected from students has been anonymised in this report.

Results and Discussion

Solving Equations using Algebra Tiles and Bar Models

Two lessons were planned collaboratively with the first using Algebra tiles (manipulatives) and the second Bar models (representations) to teach the solving of linear equations.

In lesson 1, students were introduced to algebra tiles, what they stood for, and their use in representing expressions and equations, discussing which equations were the same, equivalent or different. This was followed by a matching activity, with students linking pictorial representations of algebra tiles to equations and grouping equivalent sets. Later students used algebra tiles to first build up, then break down (solve) equations using algebra tiles.

In lesson 2, students were introduced to bar models as representations of 1 and 2-step equations. This was then linked with an algebraic "balance method" by modelling both approaches simultaneously. Students were then able to develop their understanding with scaffolded exercises given both the algebraic equation and bar models.

Teacher reflections

Student engagement with the algebra tiles was varied, with some very resistant to them and others valuing them and using them extensively. They were seen to be helpful in consolidating students understanding of "like" terms and for simplifying expressions as well as supporting student discussion. As the vast majority of students had not used algebra tiles before, some time had to be devoted to gaining a basic understanding of them, and the concept of "zero pairs" was found to be necessary when using algebra tiles to solve equations.

As with the algebra tiles, there was some resistance to the use of bar models, particularly from those students already adept at solving 2-step equations algebraically. Likewise, some lesson time had to be allocated to developing student's basic understanding of bar models before they could be used to strengthen conceptual understanding or be linked with algebraic approaches. Those students who were happy working with bar models were able to use them to solve a range of two-step equations after some initial scaffolding and showed more confidence with using the balance method alongside it. Some students who were happy with the balance method but became stuck when equations had x's on both sides were able to move forward with their own method once they had been encouraged to create their own bar model of the equation.

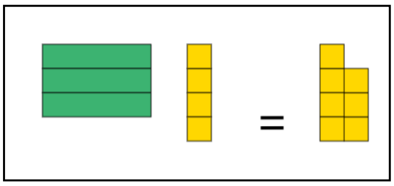
Pre-Teach Results

Equation	% Correct
4t =24	95.6
3x+5=11	85.9
2y-5=16	79.7

Students were given three equations to solve that increased in difficulty and this is reflected in the percentage of correct answers.

There was quite a difference in the students that took part in the pre and post-teach quiz so direct comparisons are not useful to make.

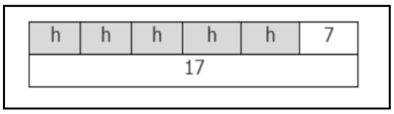
Post-Teach Results



What equation do these algebra tiles represent?

	3x+4=7	3+4x=7x	3x+4y=7y	3+4=7	% Correct
%	73.1	15.4	0.0	11.5	73.1

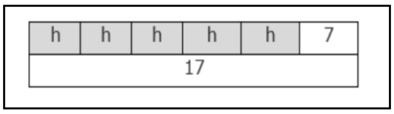
Whilst many students were able to identify the correct equations, a group of students needed more time understanding what each tile represented.



Which equation does this bar model represent?

	5h+7=17	5h-7=17	7h=17	h+7=17	% Correct
%	92.3	3.8	3.8	0.0	92.3

The vast majority of students can see the link between the visual representation and the algebra equation.



Using the bar model, solve to work out h

	2	10	3.4	4.8	% correct
%	76.9	15.4	7.7	0.0	76.9

A smaller but still large proportion of students could solve the equation from the representation.

Solve the following equation: 2x - 5 = 16

	10.5	21	11	13	% correct
%	70.8	4.2	20.8	4.2	70.8

A slightly smaller proportion were able to solve this equation where no representation was given. As this was an online quiz we were not able to see which students had attempted to draw a bar model to help them.

Speed Using Double Number Lines

Pre and Post-Teach Results

Students were asked to answer questions on Speed, Distance, Time prior to introducing them to the new model of double number lines, in order to test their initial understanding.

They were then re-tested on similar questions a week or two afterwards and the results are shown in the tables below.

Q1) Which of these is a unit of speed? Hr/Km or miles / min

Q2) A vehicle travels 2700m in 90 seconds. What is its speed in metres per second?

Q3) A spider travels at 4 cm/s. How long does it take the spider to travel 12cm?

Q4) A car is travelling on average 24 miles per hour. How far will it travel in 240 minutes?

Only three teacher's results were collected due to Covid issues.

There was an improvement of the post-teach results over the pre teach on all four questions which indicate that the double number lines model had a positive effect on student understanding.

There was a significant average improvement of over 40% over the pre-teach results on Q2 and Q4

Table 1

Pre-Teaching - Distance Time Speed Topic					
Name	**Q1**	**Q2**	**Q3**	**Q4**	**Average**
Teacher 1	63	37	47	37	46
Teacher 2	25	26	30	10	23
Teacher 3	72	71	67	44	63
Average	53	45	48	30	44

Table 2

Post Teaching - Distance Time Speed Topic					
Name	**Q1**	**Q2**	**Q3**	**Q4**	**Average**
Teacher 1	50	75	50	63	59
Teacher 2	52	90	86	67	74
Teacher 3	80	80	80	86	81
Average	61	82	72	72	71

Teacher reflections and student feedback

This was the first time that many classes had seen Double Number Lines as a representation so some teachers did pre-activities to help students understand how they worked.

Some teachers said they had an initial concern on how to deal with introducing the model where there where students in the class who felt confident using the speed formula. These teachers worked through the model questions using both methods and felt this helped value student's prior knowledge but also gave them access to DNLs as a potential support if they got stuck or wanted to check their working.

It was felt by all teachers that most students engaged with the activities and where able to successfully match the exam questions with the DNL pictures. Student discussions on this activity appeared to deepen their understanding of what speed as a compound measure is and how it links distance and time, for example that 40mph means "in 1 hour you travel 40 miles". This was borne out when students attempted exam style questions at the end even if they returned to using the formula.

Students who did use the DNLs fed back that it helped them break the question into smaller steps. Students who said they would normally miss out speed questions felt they could attempt the questions with some confidence now. “I don’t like speed questions but this makes it a bit easier”

There was some evidence of an intuitive understanding of proportional relationships e.g students were able to state that an object travelling at 40mph will travel 20 miles in 30 minutes without any calculation.

Some students worked on an activity on density which meant applying the same ideas in a different situation. Just under a half were using DNLs, a quarter using formulas and some trying to work out what to do in their head. Most of the DNLs and some formula students were being successful.

Overall students responses varied greatly from enjoying using the DNLs and finding them useful to some finding them confusing and not sure about remembering them in an exam. Classes that had used them previously were generally more positive.

Percentages using Double Number Lines

All staff had previously covered percentages with their class earlier in the year so this was delivered as a revision lesson in March. Each teacher taught the following lesson having given the students a 5 question online pre-teach test.

1) PowerPoint modelling how to use Double Number Lines to answer questions on the following topics covering.

- Percentage of amount

- Decrease by a percentage

- Increase by a percentage

- Amounts as percentages

- Reverse percentages

2) Matching activity with a similar range of percentage questions and partially completed DNLs. There was one blank questions and one blank DNL that students had to fill in to complete the matching.

3) Same Surface Different Depth activity with four exam style questions using the same numbers but asking different percentage questions

Students then completed a written 5 question post-teach test on some of the percentage topics

Pre-teach test results

Students gave an average confidence with percentages as 6.3 on a 1 to 10 scale, although this varied by class from the lowest average of 3.5 up to the highest of 8.4. This shows the variety in the ability of the classes.

Students struggled with percentage decrease but mainly because they calculated the percentage correctly but didn’t take it away. They also found writing an amount as a percentage more difficult. However there was great variation between groups for each question.

Percentage question	Average percentage correct
1) Percentage of amount	87.1
2) Decrease by a percentage	64.3
3) Increase by a percentage	71.4
4) Write an amount as a percentage	61.4
5) Reverse percentage	71.4

Post-teach test results

For this analysis we looked at whether students attempted the question using a DNL or not and if they got the question correct or not. A direct comparison of correct responses would have been confused by the different type of test and which method students were using.

Question	Used a DNL - %	Didn't use a DNL- %	Blank-%
Percentage of an amount	52.6	46.2	1.3
Increase by a percentage	53.2	35.4	11.4
Decrease by a percentage	74.2	21.1	4.7
Reverse percentage	68.4	21.5	10.1
Finding an original amount from the %/amount change	60.0	28.8	11.3

It can be seen that students were more likely to use a DNL for questions 3-5 which were (perceived) to be more difficult.

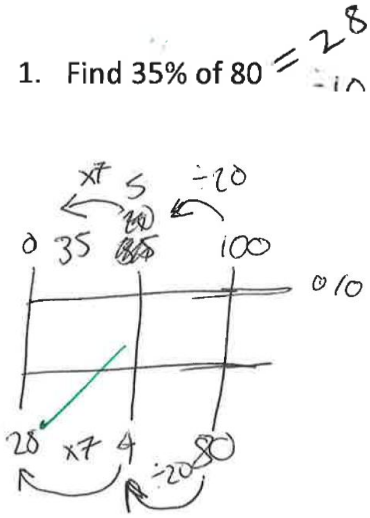
Question	Correct with DNL	Correct with no DNL
Percentage of an amount	80.5	86.1
Increase by a percentage	83.0	71.4
Decrease by a percentage	71.6	92.6
Reverse percentage	72.2	47.1
Finding an original amount from the %/amount change	41.7	26.1

Whilst there were high percentages correct for both DNL and non-DNL methods for Q1-3 , students achieved much higher success using the DNL for the "harder" Q4 and 5.

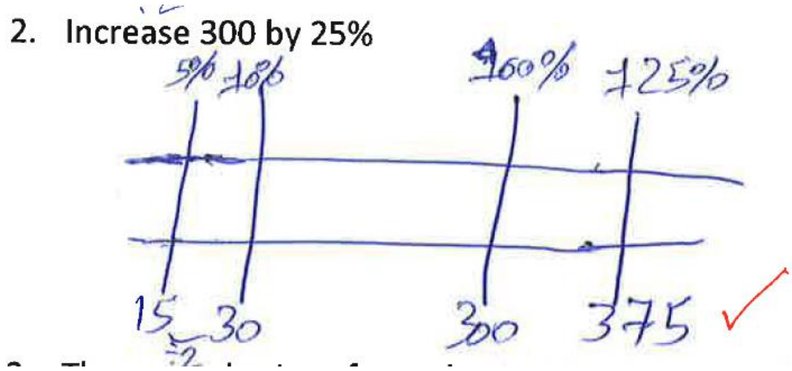
These two tables seem to indicate that students are happy to use their previous methods when they feel confident with them but are happy to move to using DNLs when the questions when they are less confident about remembering and using a previous method. Beyond this, they also achieved a higher rate of success using the DNL when it comes to reverse type percentage questions.

Student work

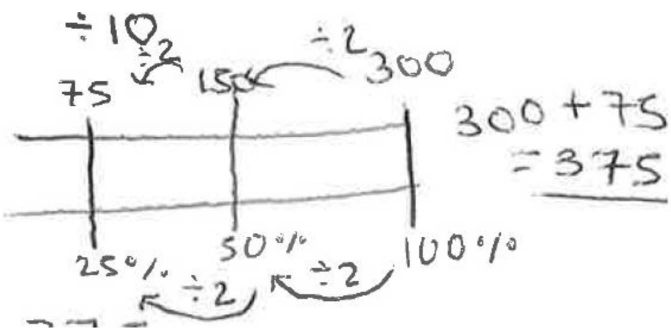
Most students but not all have drawn the two lines as part of the DNL. Many students have not added in the 0 line. There is real mix of those who wrote the scaling factors in and those who didn't as well as the use of labels for the lines. All students multiplied or divided along the line and not between the lines. The minority of students drew the arrow to show which way their scaling factor was operating.



This student has used a two step process but moved in one direction along the line and hasn't worried about where on the scale the 35% should go.

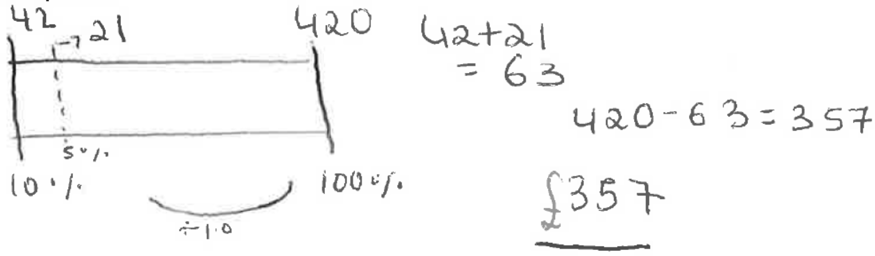


This student has found the small percentage blocks they know they need 5% and 10%, but done the final calculations in their head and generally not labelled their scaling factors.



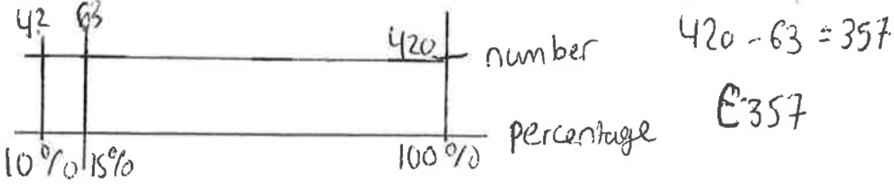
A divide by 2 twice method to find 25%

3. The normal price of a cooker is £420. In a sale, the price is reduced by 15%. What is the sale price?

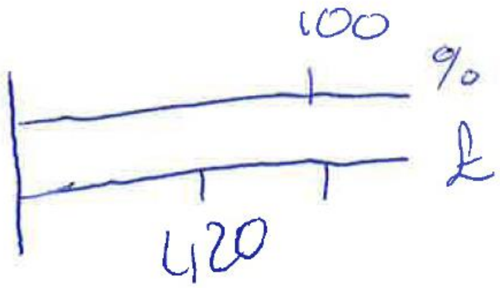


Again this students has found 10% and 5% and then done separate working and has not worried about where on the scale the 5% comes.

3. The normal price of a cooker is £420. In a sale, the price is reduced by 15%. What is the sale price?

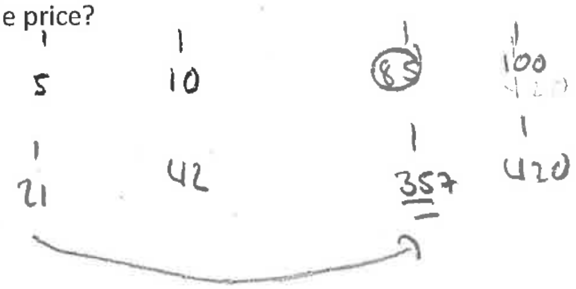


Here the student has used the line to show their thinking process but not to show the calculations



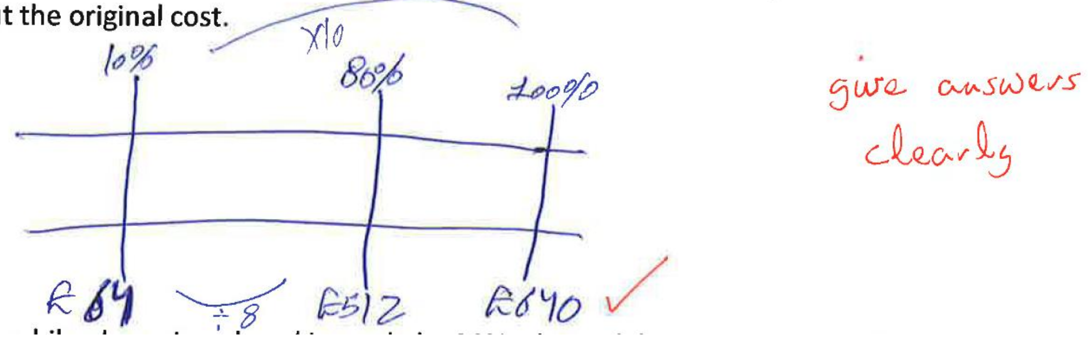
This student is stuck as they are unclear about where to put the information from the question.

3. The normal price of a cooker is £420. In a sale, the price is reduced by 15%. What is the sale price?



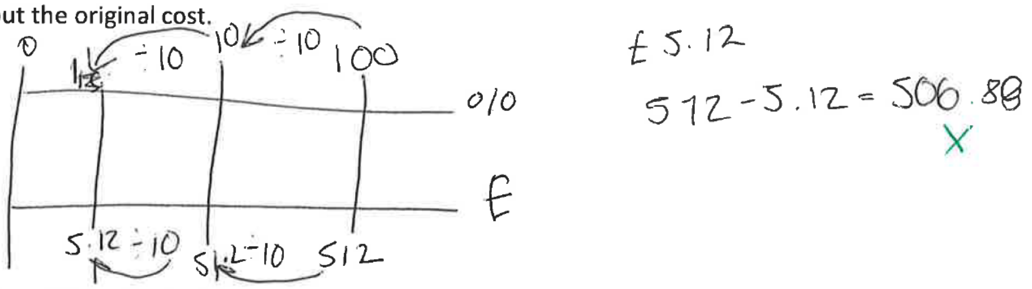
Some clear steps but they did feel the need to draw the lines. Again the steps are shown but not the working.

4. A washing machine has been reduced in a sale by 20%. The sale price is £512. Work out the original cost.



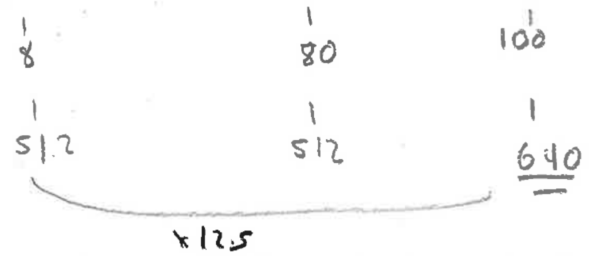
Clear steps but no arrows on the scaling

4. A washing machine has been reduced in a sale by 20%. The sale price is £512. Work out the original cost.



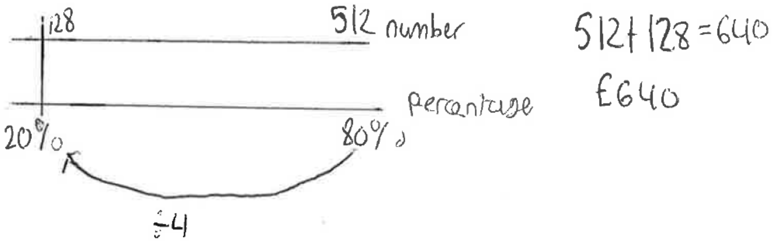
Among other things this students seems to be confused between finding 1 % and 20% but their steps are clear

4. A washing machine has been reduced in a sale by 20%. The sale price is £512. Work out the original cost.



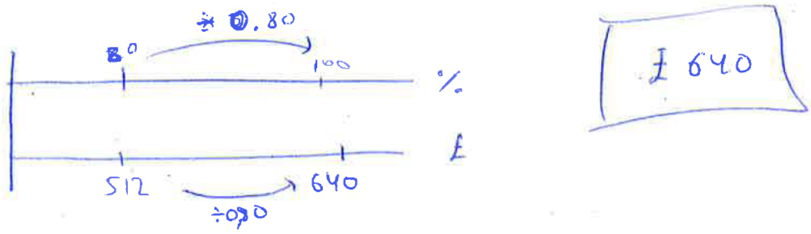
It is interesting that this student went from 80% down to 8% and then up to 100%

4. A washing machine has been reduced in a sale by 20%. The sale price is £512. Work out the original cost.



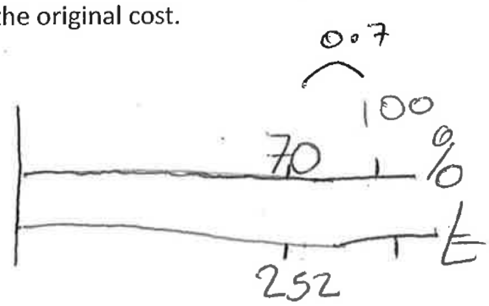
A different method shown here to work out the original 100%

4. A washing machine has been reduced in a sale by 20%. The sale price is £512. Work out the original cost.



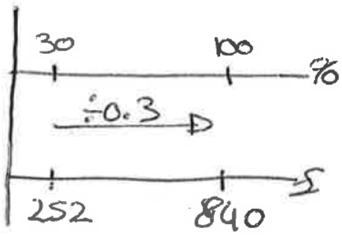
This student has used a divider rather than a multiplier.

5. A mobile phone is reduced in a sale by 30%. The mobile phone is reduced by £252. Work out the original cost.



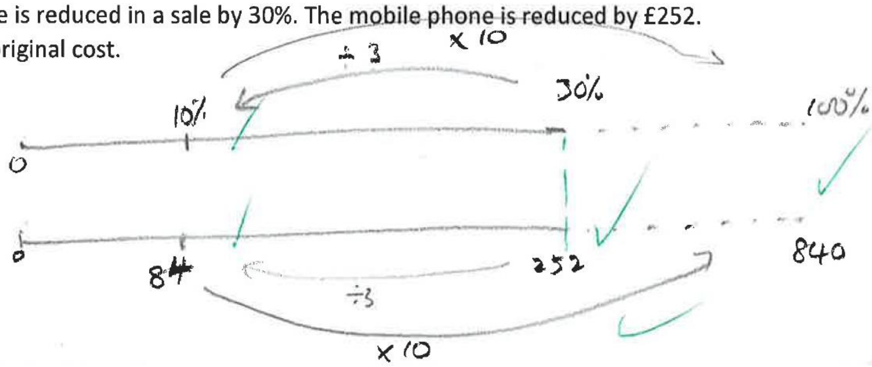
This student seems to know how to get from 70 to 100 but not what to do with the 252. Is this because they haven't labelled the 0.7 as a divide or multiply or because they have mislabelled the 252 as 70% and so won't be getting the answer other students will get?

5. A mobile phone is reduced in a sale by 30%. The mobile phone is reduced by £252. Work out the original cost.



A correct divider method

5. A mobile phone is reduced in a sale by 30%. The mobile phone is reduced by £252. Work out the original cost.



A two-step method, where they have extended the DNL during their working.

Teacher Reflections

This lesson was more of a revision lesson than the other lessons and previously some teachers had taught these topics without reference to DNL's

While there was a wide variation in the levels of classes and the amount of previous use of DNL between different teachers there were still some common findings in their reflections.

In general, most students found the DNL's helpful for visualising problems though they needed some extra support in representing an amount as a percentage and reverse percentage questions. Once they had successfully filled in the DNL there was a good understanding of using it to find the solution.

It was found that having used DNL's for simpler questions it then enabled students to progress more easily into the concept of multipliers and enabled many students to progress on to reverse percent questions who previously would have struggled with this concept. Using DNL's can help to make a smooth transition between percentage of an amount, percentage increase/decrease and reverse percentages as essentially, they are all being represented and solved in a similar way.

In some groups there were students who were resistant to using DNL's particularly with those who were already confident with other methods. With stronger groups this could be partially reduced by showing how they could help with some tricky reverse percentage questions.

In summary, DNL's helped the majority of students to visualise problems and provided a tool that enabled easier progression to harder questions. This could be more effective if introduced earlier and planned for at the start of the year.

Conclusions and Recommendations

Conclusions

Within **percentages,** representations gave weaker students a model for accessing the topic and range a questions. For stronger students they helped them make sense of and then complete more complicated questions such as reverse percentages without having to learn a new algorithm.

Using DNLs to teach **speed** helped all students develop a greater understanding of what a compound measure means and even those who continued to use the formula with questions were more able to explain why.

We only used bar models with **equations** and they were most successful in supporting students to have a greater understanding of the **balance method** rather than as a method themselves.

Similarly algebra tiles were very useful to support students who needed a more **concrete understanding** of what algebraic expressions and equations actually mean.

Recommendations

Representations can be used to support learning for many students.

- To **access** the topic, for students without a method they can confidently use.
- To take on **more challenging/varied** questions, for students who already have a method

It is better to use these representations from the **start** of the year and across a range of topics that have the same mathematical structure. This makes it efficient for teachers to allow time to introduce them properly to students and for students to develop more confidence with the models.

The representations need to be introduced to students in a **flexible** way to take into account what they already know and how well they can apply their own methods. This will encourage students to use these models when their own methods break down.

Double number lines were the most successful of our interventions and had the greatest flexibility in terms of topics they can be used with. This would make them a good representation for a department to invest time introducing and using across the year.

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NCETM KS3 Guidance: Algebra Tiles

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Appendix/Appendices

Solving Equations – Algebra Tiles

https://forms.gle/21Ytj65Ynp7e7DHr9 - Pre-teach Quiz

Solving Equations Algebra Tiles – PowerPoint

Set 1 picture cards – Matching activity, x on one side

Set 2 picture cards- Matching Activity, x on both sides

https://mathsbot.com/manipulatives/tiles - Online website to demonstrate the use of Algebra Tiles

http://www.printable-math-worksheets.com/support-files/algebra-tiles.pdf - printable algebra tiles

Solving Equations- Bar Models

Solving Equations Bar Method- AR – PowerPoint

Ex 1+2(scaffold).-Equations-using-the-bar-method--student-support-sheets

Ex1+2.-Equations-using-the-bar-method

https://forms.gle/esP83RZND7FVqXCi8 - Post-teach Quiz

Speed

https://forms.office.com/r/Gy4YvwrL0S - Pre-teach Quiz

Speed powerpoint

Matching Activity -Exam Questions on Speed

Matching activity- Double Number Lines

https://forms.office.com/r/cEhsPBrsCZ - Post-teach Quiz

Percentages Review lesson

Pre-teach Quiz

Reviewing Percentage Skills – Power point

Matching activity -percent questions

Matching Activity - Double number lines

SSDD Activity – Same Surface Different Depth Activity

Post Teach Quiz (Percentages)